An Upper Bound for the Solvability Probability of a Random Stable Roommates Instance

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ABSTRACT

It is well-known that not all instances of the stable roommates problem admit a stable matching. Here we establish the first nontrivial upper bound on the limiting behavior of P_n, the probability that a random roommates instance of size n has a stable matching, namely, $\overline{\lim}_{n\to\infty} P_n \le e^{1/2}/2$ (=0.8244 . . .).

Key Words: stable matching, partition, proposal algorithm, random preferences, probabilistic analysis

1. INTRODUCTION

An instance of size n of the stable roommates problem consists of n persons, n even (for the moment), each of whom ranks all of the others in strict order of preference. A matching (a partition of the persons into pairs) is *unstable* if there are persons x and y who are not paired with each other, but each of whom prefers the other to his partner in the matching. Such a pair is said to *block* the matching. A matching for which there are no blocking pairs is called *stable*.

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This problem generalises the better-known stable marriage problem, in which the persons form two equal-sized disjoint sets, the men and the women, and each person ranks the members of the opposite sex. In this case a matching consists of man–woman pairs, with the concept of stability defined analogously.

Gale and Shapley [3] showed that at least one stable matching exists for every instance of the stable marriage problem, but that some stable roommates instances admit no stable matching.

Example. The following roommates instance of size 6, in which the persons are numbered $1, \ldots, 6$ and the preference lists are set out horizontally, admits the unique stable matching $\{1,2\}$, $\{3,4\}$, $\{5,6\}$:

1:	4 6 2 3 5
2:	6 1 5 3 4
3:	2 1 6 5 4
4:	3 6 5 2 1
5:	4 1 2 6 3
6:	5 4 3 1 2

By contrast, it is easily verified that the following instance of size 4 admits no stable matching;

1:	2 3 4
2:	3 1 4
3:	1 2 4
4:	3 2 1

Here, anyone who is paired with person 4 is bound to cause instability. ∎

Also in [3], Gale and Shapley described an efficient algorithm to determine a stable matching for a stable marriage instance. Much later, Irving [5] gave an efficient algorithm to find a stable matching in the roommates case, or to show that no stable matching exists. Gusfield and Irving [4] present a comprehensive study of both problems.

The question arises [4, Open Problem 8] as to the probability P_n that a random roommates instance of size n is solvable (i.e., admits at least one stable matching), and in particular as to the limiting behavior of P_n as n grows large. Empirical evidence is presented in [4] to suggest that P_n decreases as n grows, but this evidence is not conclusive in suggesting whether P_n is bounded away from zero. Very recently, Pittel [9] proved that the expected number of stable matchings is asymptotic to $e^{1/2}$ (thus bounded) and that $P_n \gtrsim (4e^3/\pi n)^{1/2}$, so if $\overline{\lim}_{n\to\infty} P_n = 0$, the rate of convergence is quite low.

It is not hard to construct an unsolvable instance of any given size $n \geq 4$, so that $P_n < 1$ for all $n \geq 4$, but hitherto no bound less than 1 has been established on P_n. It is our objective here to establish the first such bound, namely, $\overline{\lim}_{n\to\infty} P_n \leq e^{1/2}/2$. The empirical evidence of [4], extended in Section 5 of the present paper, suggests that this upper bound is by no means best possible, and this result, and the lower estimate from [9], should be seen merely as first steps towards establishing the true limiting behavior of P_n. However, the existence of such a bound is enough to illustrate a dramatic difference between the stable roommates and stable marriage problems since, for the latter, Pittel [11] showed that the likely number of solutions is at least $n^{1/2+o(1)}$.

The rest of the paper is structured as follows. In Section 2, we describe the key idea of a stable partition, first introduced by Tan [13]. This concept provides us with a neat characterization of roommates solvability. In Section 3, we examine the dynamic roommates algorithm, also due to Tan [12], which enables us to identify a certain structure called a *core configuration*, which must be present in any solvable roommates instance. In Section 4, we study the expected number of these core configurations, and derivation of the limiting value of this number enables us to bound from above the solvability probability P_n. Finally, in Section 5, we summarize the latest empirical evidence, which suggests that there is still a considerable gap between this bound and the true limiting behavior of P_n, while still leaving us tantalizingly short of convincing support for any conjecture.

2. STABLE PARTITIONS

In [4, Open Problem 10], Gusfield and Irving asked whether it is possible to provide a succinct "certificate" for an unsolvable roommates instance, in the same sense as a stable matching, which can easily be verified for stability, provides a succinct certificate of solvability. This question was answered in the affirmative by Tan [13], who introduced the notion of a stable partition, showed that every roommates instance admits at least one such partition, and that an instance is unsolvable if and only if it admits a stable partition with one or more subsets of odd cardinality.

From now on, we allow roommates instances of odd as well as even size. So a roommates instance is specified by a positive integer n, and for each i $(1 \le i \le n)$ a permutation P_i of the set $\{1, 2, \ldots, n\}$ in which i itself occupies position n $[P_i(n) = i]$. The permutation P_i constitutes the *preference list* of person i— $P_i(k) = j$ if person j occupies position k in the preference list of person i; it is convenient to add each person to the end of his own preference list. Alternatively and equivalently, the instance could be specified by the *ranking list* R_i of each person i, defined as the inverse permutation of P_i.

For a given roommates instance, a *stable partition* (perhaps more properly called a *stable permutation*), is a permutation Π of the n-set $\{1, 2, \ldots, n\}$ of persons such that;

(i) for every i, $R_i(\Pi(i)) \le R_i(\Pi^{-1}(i))$;
(ii) if $R_i(j) < R_i(\Pi^{-1}(i))$, then $R_j(i) > R_j(\Pi^{-1}(j))$.

Viewing this permutation in terms of its cyclic decomposition, we refer to $\Pi(i)$ and $\Pi^{-1}(i)$ as the *successor* of i and the *predecessor* of i, respectively, in the partition. In these terms, condition (i) states that no one prefers his predecessor to his successor, and condition (ii) states that if i prefers j to his own predecessor, then j prefers his own predecessor to i.

Note that i may be both his own successor and predecessor, if he is a fixed point of Π, and i and j may be both the successor and predecessor of each other, if (i, j) forms a transposition in Π—in this case we say that $\{i, j\}$ forms a pair in the partition. In general, of course, j is the successor of i if and only if i is the predecessor of j.

It should be clear at once from the stability condition (ii) in the definition of a stable partition that the special case in which all cycles of the permutation have length 2 is precisely a stable matching for the roommates instance in question. Just as in the special case of a stable matching, we say that $\{x, y\}$ forms a *stable pair* and x and y are *stable partners* if they form a pair in some stable partition.

Example. The following instance of size 6 has 5 stable partitions, 3 of which are stable matchings:

1:	2 4 3 6 5 1
2:	6 5 4 1 3 2
3:	2 5 6 1 4 3
4:	5 2 1 3 6 4
5:	1 3 2 4 6 5
6:	3 1 4 5 2 6

The stable partitions are: $\Pi_1 = (1\ 4\ 2\ 6)(3\ 5)$, $\Pi_2 = (1\ 6\ 3\ 5)(2\ 4)$, $\Pi_3 = (1\ 4)(2\ 6)(3\ 5)$, $\Pi_4 = (1\ 6)(2\ 4)(3\ 5)$, $\Pi_5 = (1\ 5)(3\ 6)(2\ 4)$. ■

Example. This second instance of size 6 has a unique stable partition containing two odd-length cycles, and is unsolvable:

1:	2 3 6 5 4 1
2:	6 1 3 4 5 2
3:	6 2 5 1 4 3
4:	6 2 5 1 3 4
5:	1 2 3 6 4 5
6:	5 2 1 3 4 6

The only stable partition in $\Pi = (1\ 3\ 5)(2\ 6)(4)$. ■

The following significant results were established by Tan [13].

Theorem 2.1. *Every roommates instance admits at least one stable partition.*

Theorem 2.2. *If* Π *is a stable partition for a given roommates instance, and* $C = (a_1, a_2, \ldots, a_{2m})$ $(m \geq 2)$ *is an even length cycle of* Π, *then replacing* C *by the transpositions* $(a_1, a_2), \ldots, (a_{2m-1}, a_{2m})$, *or by the transpositions* $(a_2, a_3), \ldots, (a_{2m}, a_1)$, *gives another stable partition.*

We will call a stable partition in which all even-length cycles are of length 2 a *reduced* stable partition. As a corollary of Theorem 2.2, it follows that an instance admitting a stable partition in which all cycles are of even length is a solvable instance.

Theorem 2.3. *If C is an odd-length cycle in a stable partition for a given roommates instance, then C is a cycle in all stable partitions for that instance.*

It follows from Theorem 2.3 that an instance that admits a stable partition with an odd-length cycle is unsolvable, and from the above three theorems that such a stable partition can serve as a succinct certificate of unsolvability. (Clearly, and trivially, every roommates instance of odd size is unsolvable.) It also follows from Theorem 2.3 that all stable partitions for a given roommates instance contain the same number of odd-length cycles—we shall refer to this as the *odd-cycle index* of the instance.

Note that the concept of an odd-length cycle in a stable partition, and the unsolvability of any instance admitting such a cycle, was implicit in the notion of an "improper rotation" introduced by Irving [6], but the full significance of the concept was first understood and established by Tan [13].

In the terminology of Gusfield and Irving [4, Chapter 4], for a solvable roommates instance, a stable partition corresponds to a stable table after all singular rotations have been eliminated, and in which the non-singular rotation ρ is exposed if and only if its dual $\bar{\rho}$ is also exposed.

3. THE DYNAMIC ROOMMATES PROBLEM

In [12], Tan addressed the following problem: Given a roommates instance of size n and a stable partition for that instance, how can we find a stable partition for the instance of size $n+1$ arising from the arrival of a new person?

To be more precise, an instance I_{n+1} of size $n+1$ is created from a given instance I_n of size n in the following way:

(1) Person $n+1$ is added, with preference list a permutation P_{n+1} of $\{1, 2, \ldots, n+1\}$ in which $P_{n+1}(n+1) = n+1$.
(2) For each person i $(1 \le i \le n)$, person $n+1$ is inserted at some position j $(1 \le j \le n)$ in i's preference list, and all persons occupying positions k $(j \le k \le n)$ are demoted by one position.

It turns out, as shown in [12], that a stable partition for I_{n+1} can be found from a stable partition for I_n by application of a "proposal sequence" reminiscent of the algorithms of Gale and Shapley [3] for the stable marriage problem and Irving [5] for the roommates problem. This proposal sequence starts from a (reduced) stable partition for I_n and the new "free" person. Each proposal except for the last leads to a single change in the current partition, and in the identity of the free person, until the last proposal completes the derived (reduced) partition for the extended instance.

Given a stable partition Π for a roommates instance of size n, Tan's algorithm for a stable partition Π' in the extended instance of size $n+1$ is described in Figure 1. For this version of the algorithm, it is assumed that Π is a reduced stable partition, a valid assumption in view of Theorem 2.2. In fact, as described, the algorithm transforms Π from a reduced stable partition for the original instance into a reduced stable partition for the extended instance. Note also that we

```
p := the new person n + 1 ; {throughout, p is the "proposer"}
repeat
    if such a q exists then
        q := first person on p's list who prefers p to his own predecessor in Π
    else
        q := p ; {q is the "proposee"}
    if q = p then
        begin
        Π(p) := p ;
        exit {with p forming a cycle of length 1 in Π}
        end
    else
        if q was in an odd-length cycle C of Π then
            begin
            in Π match up other members of C in successive pairs;
            pair off p and q;
            exit
            end
        else
            if q was previously a proposer then
                exit with a new odd cycle in Π
                    containing all proposees and proposers since q
            else {q must be currently matched}
                begin
                t := Π(q); {t is "jilted" by q}
                pair off p and q;
                p := t {the jilted person becomes the new proposer}
                end
forever
```

Fig. 1. The dynamic roommates algorithm.

express the algorithm in a form in which a proposal that would be rejected is simply not made; at each stage, the current proposer proposes to the person he likes best among all those who prefer him to their predecessor in the current partition, or to himself if there is no such person.

In essence, the proposal sequence of Tan's algorithm, initiated by the new person, continues until one of three terminating conditions arises:

(a) The proposer p proposes to himself; in this case the algorithm terminates with p forming a cycle of length 1 in the stable partition for the extended instance.
(b) The proposer p proposes to a person q who is in an odd-length cycle $C = (q, x_1, \ldots, x_{2m})$ of Π; in this case the algorithm terminates with p and q forming a transposition, and with x_{2i-1} and x_{2i} $(i = 1, \ldots, m)$ also forming transpositions in the derived stable partition;
(c) The proposer p proposes to a person q who was previously himself a

proposer; in this case the algorithm terminates with an odd-length cycle $C = (q, p, x_{2m-1}, x_{2m-2}, \ldots, x_1)$, where the proposal sequence since q himself proposed was $q \to x_1$, $x_2 \to x_3, \ldots, x_{2m-2} \to x_{2m-1}$, and $(x_{2m} =) p \to q$.

If p's proposal to q leads to none of cases (a), (b), (c), then, because Π is reduced, q must currently be in a transposition, or, in other words, q has a partner; instead, p and q become partners, and q's ex-partner becomes the new proposer.

Further details of the algorithm and a proof of its correctness may be found in [12].

Example. Consider the following instance of size 7 and odd-cycle index 1, where we display a reduced stable partition (2 3 6)(1 5)(4 7) in skeleton preference lists.

```
1:  . . . 5 . . 1
2:  . 3 . . 6 . 2
3:  . 6 . . 2 . 3
4:  . . . 7 . . 4
5:  . . . 1 . . 5
6:  . 2 . . 3 . 6
7:  . . . 4 . . 7
```

Along comes person 8 with preference list

```
8:  1 7 2 5 3 6 4 8
```

As an example of case (a), suppose that the (suitably expanded) preference lists are

```
1:  . . . 5 . 8 4 1
2:  . 3 . . 6 4 . 2
3:  . 6 . . 2 4 . 3
4:  1 2 3 7 6 8 5 4
5:  . . . 1 . 4 . 5
6:  . 2 . . 3 . 4 6
7:  . 8 . . 4 . . 7
8:  1 7 2 5 3 6 4 8
```

Then the proposal sequence is: $8 \to 7$, $4 \to 4$, and we exit with the stable partition (2 3 6)(1 5)(7 8)(4), the extended instance having odd-cycle index equal to 2.

As an example of case (b), suppose that the (suitably expanded) preference lists are

1:	.	.	.	5	7	8	.	1
2:	.	3	.	.	6	8	.	2
3:	.	6	.	.	2	.	.	3
4:	.	.	.	7	5	6	.	4
5:	.	8	4	1	.	.	.	5
6:	.	2	4	.	3	.	.	6
7:	1	8	.	.	4	.	.	7
8:	1	7	2	5	3	6	4	8

Then the proposal sequence is: $8 \to 7$, $4 \to 5$, $1 \to 7$, $8 \to 5$, $4 \to 6$; and since 6 is in an odd-length cycle, the extended instance has odd-cycle index 0, and we exit with the stable partition (stable matching in this case) (1 7)(8 5)(4 6)(2 3).

Finally, as an example of case (c), suppose that the (suitably expanded) preference lists are

1:	.	.	.	5	8	.	.	1
2:	.	3	.	.	6	.	.	2
3:	.	6	.	.	2	.	.	3
4:	.	.	.	7	5	.	8	4
5:	.	.	4	1	8	.	.	5
6:	.	2	.	.	3	.	.	6
7:	.	8	.	.	4	.	.	7
8:	1	7	2	5	3	6	4	8

Then the proposal sequence is: $8 \to 7$, $4 \to 5$, $1 \to 8$, and since 8 was already a proposer, the extended instance has odd-cycle index 2, and we exit with the stable partition (8 1 5 4 7)(2 3 6). ■

As observed by Tan, it is an immediate consequence of this algorithm and Theorem 2.3 that if I_{n+1} is an instance of size $n+1$ obtained by adding a new person to a given instance I_n of size n, then the odd-cycle index of I_{n+1} differs by exactly one from the odd-cycle index of I_n. In particular, if I_{n+1} is a solvable instance, then I_n must have odd-cycle index 1, and application of Tan's algorithm must terminate in case (b) above. It is now our objective to show that, in addition, in this case, the reduced stable partition in I_n relative to which the proposal sequence is as short as possible has some additional properties. Since these properties are crucial to the analysis in the next section, we state and prove them formally as a theorem.

Theorem 3.1. *Let I be a solvable roommates instance of size n (n even) and let I_m (of odd-cycle index 1) be the instance of size $n-1$ obtained by deleting person m ($1 \le m \le n$). Let Π be a reduced stable partition for I_m relative to which the proposal sequence in Tan's algorithm is as short as possible, say $m \to q_1$, $p_1 \to q_2, \ldots, p_{r-1} \to q_r$, where q_r is in the odd-length cycle. Then*

(*i*) *the sequence p_0 ($=m$), $q_1, p_1, q_2, \ldots, p_{r-1}, q_r$ consists of $2r$ distinct persons;*

(*ii*) *there is no pair* p_i, q_j $(1 \le j < i \le r-1)$ *such that* p_i *prefers* q_j *to* q_{i+1} *and* q_j *prefers* p_i *to* p_j.

Note: It is true of every proposal sequence that there is no pair p_i, q_j $(1 \le j < i \le r-1)$ such that p_i prefers q_j to q_{i+1} and q_j prefers p_i to p_{j-1}. However the condition (ii) is stronger since q_j prefers p_{j-1} to p_j. Put differently, (i) and (ii) imply that q_{i+1} is the best choice for p_i among all members (different from m) who prefer p_i to their predecessor in Π.

Proof. (i) $p_i = q_i$ and $p_i = q_{i+1}$ are trivially impossible. It is not possible that $p_i = q_j$ for some i, j with $i+1 < j$, for this would imply that I has odd-cycle index 2, and therefore is unsolvable [see case (c) of Tan's algorithm]. Further, if $p_i = q_j$ for some i, j with $i > j$ and i, j as small as possible, then it is immediate that $q_i = p_{j-1}$, and we reach the same conclusion.

Suppose that the q's are not all distinct, and that the first repeated entry in the sequence of q's is $q_i = q_j$ with $1 \le i < j \le r-1$. (Note that $j \le r-1$ since q_r is the first occurrence in the sequence of a person in the odd-length cycle.) Then it follows that $p_{i-1} = p_j$, since when q_j $(=q_i)$ received the proposal from p_{j-1}, he (q_j) was paired with p_{i-1} —so p_{i-1} $(=p_j)$ became the free person at that point.

It suffices therefore to show that all the p's are distinct. Suppose not, and that the first repeated entry in the sequence of p's is $p_i = p_j$ with $0 \le i < j \le r-1$. Then $\{p_1, q_1\}, \ldots, \{p_{j-1}, q_{j-1}\}$ are distinct pairs in Π. Also, $q_j = q_{i+1}$, since when q_j receives a proposal, he is paired with p_j $(=p_i)$. We claim that replacing the pairs $\{p_{i+1}, q_{i+1}\}, \ldots, \{p_{j-1}, q_{j-1}\}$ by the pairs $\{p_{i+1}, q_{i+2}\}, \ldots, \{p_{j-1}, q_j\}$ yields another stable partition Θ for I_m, and that the proposal sequence relative to Θ is $m \to q_1, \ldots, p_{i-1} \to q_i, p_j \to q_{j+1}, \ldots, p_{r-1} \to q_r$. This is a shorter proposal sequence than that relative to Π, giving a contradiction.

Suppose that the indicated partition Θ is not stable, and that there are persons x, y such that x prefers y to $\Theta^{-1}(x)$ and y prefers x to $\Theta^{-1}(y)$. Let $\mathcal{P} = \{p_{i+1}, \ldots, p_{j-1}\}$, $\mathcal{Q} = \{q_{i+1}, \ldots, q_{j-1}\}$. Note that, in Π, $p_k \in \mathcal{P}$ is paired with q_k; his first change of partner in the proposal sequence occurs when he proposes to q_{k+1}, his partner in Θ. Likewise in Π, $q_k \in \mathcal{Q}$ is paired with p_k; his first change of partner occurs when he receives a proposal from p_{k-1}, his partner in Θ. Any person $x \not\in \mathcal{P} \cup \mathcal{Q}$ has the same partner in Θ as in Π.

We consider three cases.

(1) $x \not\in \mathcal{P} \cup \mathcal{Q}$, $y \not\in \mathcal{P} \cup \mathcal{Q}$. Then $\Theta^{-1}(x) = \Pi^{-1}(x)$, $\Theta^{-1}(y) = \Pi^{-1}(y)$. So the instability of Θ due to x and y implies the instability of Π due to x and y.

(2) $x \in \mathcal{P} \cup \mathcal{Q}$, $y \not\in \mathcal{P} \cup \mathcal{Q}$.

 (a) $x \in \mathcal{P}$, $x = p_k$ say. Since $\Theta^{-1}(y) = \Pi^{-1}(y)$, y prefers p_k $(=x)$ to $\Pi^{-1}(y)$. Also p_k prefers y to $\Theta^{-1}(x)$ $(=q_{k+1})$, so p_k would have proposed to y instead of q_{k+1}—a contradiction.

 (b) $x \in \mathcal{Q}$, $x = q_k$ say. Then again y prefers q_k to $\Pi^{-1}(y)$. But x $(=q_k)$ prefers $\Theta^{-1}(x)$ $(=p_{k-1})$ to Π^{-1} $(x)(=p_k)$, and therefore prefers y to $\Pi^{-1}(x)$, contradicting the stability of Π.

(3) $x \in \mathcal{P} \cup \mathcal{Q}$, $y \in \mathcal{P} \cup \mathcal{Q}$. It is clear that each member of $\mathcal{P}$ prefers his partner in Π to his partner in Θ, while the opposite is true for each member of $\mathcal{Q}$. So x, y cannot both belong to $\mathcal{Q}$, for then the instability of Θ

would imply the instability of Π. If x, y both belong to $\mathscr{P}$, say $x = p_k$, $y = p_l$ with $k < l$, then when y proposes to q_{l+1} $[=\Theta^{-1}(y)]$, x is already paired with q_{k+1} $[=\Theta^{-1}(x)]$ so that x prefers y to his partner at that time. Therefore, y should propose to x instead of q_{l+1}, a contradiction.

Finally, if one of x, y is in $\mathscr{P}$ and the other is in $\mathscr{Q}$, say $x = p_k$, $y = q_l$, then x prefers $\Pi^{-1}(x)$ $(=q_k)$ to $\Theta^{-1}(x)$ $(=q_{k+1})$ and y prefers $\Theta^{-1}(y)$ $(=p_{l-1})$ to $\Pi^{-1}(y)$ $(=p_l)$. So when x proposes to $\Theta^{-1}(x)$, regardless of whether y has received a proposal from $\Theta^{-1}(y)$ or not, y prefers x to his partner, and so x should propose to y instead of q_{k+1}, again a contradiction.

As far as the proposal sequence relative to Θ is concerned, it is clear that proposals up to and including $p_{i-1} \to q_i$ are unaffected. At that point, the pairings are exactly as they would be after the proposal $p_{j-1} \to q_j$ in the proposal sequence relative to Π. So subsequent proposals are unaffected, and the proposal sequence is as claimed.

(ii) Suppose, again for a contradiction, that there are p_i, q_j $(j < i)$ such that p_i prefers q_j to q_{i+1} and q_j prefers p_i to p_j. Then, by arguments similar to those used in the proof of (i), we can show that replacing the pairs $\{p_j, q_j\}, \ldots, \{p_i, q_i\}$ by the pairs $\{p_j, q_{j+1}\}, \ldots, \{p_{i-1}, q_i\}, \{p_i, q_j\}$ yields another stable partition Θ for I_m, and that the proposal sequence relative to Θ is $m \to q_1, \ldots, p_{j-1} \to q_j$, $p_i \to q_{i+1}, \ldots, p_{r-1} \to q_r$. This would again be a shorter proposal sequence than that relative to Π, giving a contradiction. We omit the further details. ∎

Example. In the second case illustrated in the earlier example, the proposal sequence was $8 \to 7$, $4 \to 5$, $1 \to 7$, $8 \to 5$, $4 \to 6$ relative to the initial stable partition (2 3 6)(1 5)(4 7). Replacing the pairs (1 5), (4 7) by the pairs (4 5), (1 7) leads to the proposal sequence $8 \to 7$, $4 \to 6$ relative to the initial stable partition (2 3 6) (4 5) (1 7). ∎

For a solvable roommates instance of even size n, we will call a stable partition and associated proposal sequence satisfying the properties (i) and (ii) of the above theorem a *core configuration* (relative to person n). We will see that these core configurations play a crucial role in the next section. We conclude the present section with a uniqueness result for core configurations. For this we require some preliminary lemmas.

The first lemma is proved in [6, Corollary 2.4.2].

Lemma 3.1. *Let Π be a reduced stable partition for a roommates instance in which x, y form a matched pair, and let Π' be any stable partition in which x, y do not form a matched pair. Then one of x, y has a better partner, and the other a worse partner in Π' than in Π.*

Lemma 3.2. *For a given roommates instance, y is the worst stable partner of x if and only if x is the best stable partner of y.*

Proof. This is an immediate consequence of the previous lemma. ∎

Lemma 3.3. *For a given roommates instance I_n of odd cardinality n, with* *odd-cycle index* 1, *let stable partition* Π *and sequence* $p_0, q_1, p_1, q_2, \ldots, p_{r-1}, q_r$ *be a core configuration, where* p_0 *is the new person and* q_r *is a member of the odd cycle. Then, for* $i = 1, \ldots, r-1$,

(*a*) q_i *is the worst stable partner of* p_i *in* I_n;
(*b*) p_i *is the best stable partner of* q_i *in* I_n.

Proof. (b) is a consequence of (a) and the previous lemma, so it suffices to establish (a). We use backwards induction on i. For the base case, suppose that x, rather than q_{r-1}, is the worst stable partner of p_{r-1} — so p_{r-1} prefers q_{r-1} to x. Certainly p_{r-1} must prefer x to q_r, for otherwise the pair p_{r-1}, q_r would block the stable partition in which p_{r-1} and x are partners. (x cannot be equal to q_r, for q_r is in the odd-length cycle in all stable partitions of I_n.) So why did p_{r-1} not propose to x when he proposed to q_r? Since p_{r-1} is the best stable partner of x, it follows that x prefers p_{r-1} to his partner in Π. So, in the meantime, x must have received a better proposal, so x is a proposee, say $x = q_j$. But then q_j prefers p_{r-1} to p_j, and the pair p_{r-1}, q_j contradict the second defining property of a core configuration.

For the induction step, suppose that q_i is the worst stable partner of p_i, and that x, rather than q_{i-1} is the worst stable partner of p_{i-1} in I_n.

(i) p_{i-1} prefers q_i to x. Then since p_i is q_i's best stable partner, and q_i prefers p_{i-1} to p_i, p_{i-1}, q_i form a blocking pair for the supposed stable partition in which p_{i-1} is paired with x.
(ii) $x = q_i$. Then p_{i-1} is a stable partner for q_i, and q_i prefers p_{i-1} to p_i, his supposed best stable partner — a contradiction.
(iii) p_{i-1} prefers x to q_i. So why does p_{i-1} not propose to x when rejected by q_{i-1}? Since p_{i-1} is the best stable partner for x, it follows that x prefers p_{i-1} to his partner in Π. So, in the meantime, x must have received a better proposal, so x is a proposee, say $x = q_j$. But then q_j prefers p_{i-1} to p_j, and the pair p_{i-1}, q_j contradict the second defining property of a core configuration. ∎

Theorem 3.2. *All core configurations relative to person n for a given solvable roommates instance of even cardinality n have the same proposal sequence.*

Proof. In the sequence p_0 $(=n)$, $q_1, \ldots$ of any core configuration, q_1 is the first person in the list of p_0 who prefers p_0 to his best stable partner. For, by the previous lemma, q_1 certainly does prefer p_0 to his best stable partner, and if this is true of any predecessor x of q_1 on p_0's list, then p_0 would certainly propose to x in preference to q_1. So the beginning of the sequence of a core configuration is uniquely defined, and the uniqueness of the rest of the sequence follows at once from the previous lemma. ∎

4. BOUNDING THE SOLVABILITY PROBABILITY

For n an even positive integer, let P_n denote the probability that a random stable roommates instance of size n is solvable. Also, for a given instance of size n, denote by C_n the number of core configurations relative to person n. Note that, if the instance is solvable, there must be at least one such core configuration. (Although the proposal sequence associated with core configurations is unique, by Lemma 3.2, core configurations may differ in the transpositions in their reduced stable partitions.) Write $C_n = C'_n + C''_n$, where C'_n, C''_n are the numbers of core configurations with odd cycle of length ≥ 3, and equal to 1, respectively. From Pittel [8, Theorem 3.1], it follows that

$$P(C''_n = 0) \geq 1 - \exp(-cn^{1/2}) \quad (c > 0)\,.$$

So

$$P_n \leq E(C'_n) + O(\exp(-cn^{1/2}))$$

and we need to establish the following result:

Theorem 4.1. $\lim_{n\to\infty} E(C'_n) = e^{1/2}/2$.

To prove the theorem, we need to derive first an integral-type formula for the expectation $E(C'_n)$.

Let $k \geq 0$, $l \geq 3$ be two nonnegative integers such that k is even, l is odd, and $k + l \leq n - 1$. Introduce $C_n(k, l)$, the random number of all core configurations (relative to member n) with an odd cycle of length l, and a proposal sequence involving k additional members from the set $\{1, \ldots, n-1\}$ ($k = r - 1$ in the notation of Theorem 3.1). By symmetry,

$$E(C_n(k, l)) = N_n(k, l)P_n(k, l)\,.$$

Here $P_n(k, l)$ is the probability that a partition $\Pi = (1, 2), (3, 4), \ldots, (k-1, k), (k+1, \ldots, k+l), (k+l+1, k+l+2), \ldots, (n-2, n-1)$ of $\{1, 2, \ldots, n-1\}$ is stable, and is the partition component of a core configuration, while the proposal sequence component is $n \to 1, 2 \to 3, \ldots, k-2 \to k-1, k \to k+1$ [the cycle $(k+1, \ldots, k+l)$ is in the order "from member to member's predecessor"].

As for $N_n(k, l)$, it is the number of ways to choose an ordered sequence of $(k+l)$ members from $\{1, \ldots, n-1\}$, and to then pair the remaining $(n-1-k-l)$ members. Thus,

$$\begin{aligned} N_n(k, l) &= (n-1)(n-2)\ldots(n-k-l)(n-k-l-2)!! \\ &= (n-1)!/(n-k-l-1)!!\,, \end{aligned}$$

where $x!!$ denotes the product of the integers $\leq x$ that are of the same parity as x.

Lemma 4.1. *With the n-dimensional unit cube* $\mathscr{C} = \{z = (z_i)_{i=1}^n.\ 0 \leq z_j \leq 1,\ 1 \leq j \leq n\}$, *we have*

$$P_n(k, l) = \int_{z\in\mathscr{C}} \left[\prod_{\{i,j\}\notin D} (1 - z_i z_j)\right] z_{k+1}\left(\prod_{i=1}^{k+l} z_i\right) dz\,. \tag{1}$$

Here D is the set of all unordered pairs $\{u, v\}$, $1 \le u, v \le n$, such that $\{u, v\}$ is either a pair of neighboring members in the sequence $(n, 1, 2, \ldots, k, k+1)$, or in the cyclic sequence $(k+1, \ldots, k+l)$, or a pair $(\alpha, \alpha+1)$ $(k+l+1 \le \alpha \le n-2$, α even).

Proof. A random instance of the roommates problem can be generated as follows (see [8], [9]). Introduce an $n \times n$ matrix $X = [X_{ij}]$ $(1 \le i \ne j \le n)$, whose entries are independent, [0, 1]-uniform random variables. Assume that the member i ranks other members in increasing order of the elements of the ith row of X. So, for instance, his best choice, is j_0 such that $X_{ij_0} = \min_j X_{ij}$. Then clearly the ordering by i is uniformly random, and the orderings by different members are independent.

Denote by Π' the pairing $(n, 1)$, $(2, 3), \ldots, (k, k+1)$, $(k+2, k+3), \ldots$, $(k+l-1, k+l)$, $(k+l+1, k+l+2)$, $\ldots$, $(n-2, n-1)$. Let $f: \{1, \ldots, n-1\} \to \{1, \ldots, n-1\}$, $g: \{1, \ldots, n-1\} \to \{1, \ldots, n-1\}$ be the mappings for Π and Π', respectively.

Let us compute $P_n(k, l \mid x, y)$, the conditional probability of the event in question, given the values $x_i = X_{i,f(i)}$ $(1 \le i \le n-1)$ and $y_i = X_{i,g(i)}$ $(1 \le i \le n)$. Clearly $P_n(k, l|x, y) = 0$ unless x, y satisfy the conditions: for x_i, $y_i \in$ the open interval $(0, 1)$,

$$y_1 < x_1\,, \quad y_2 > x_2, \ldots, y_k > x_k\,, \quad y_{k+1} < x_{k+1}\,, \tag{2}$$

[each proposee gains, each proposer ($\ne n$) loses], and

$$y_{k+2} = x_{k+2}\,, \quad y_{k+3} < x_{k+3}, \ldots, y_{k+l-1} = x_{k+l-1}\,, \quad y_{k+l} < x_{k+l} \tag{3}$$

[each even member of the odd cycle keeps his predecessor, each odd member gets a better one]. So, assume that the conditions are met.

Turn to the conditions which involve also X_{ij} ($\{i, j\} \not\in D$). Like odd members $k+3, \ldots, k+l$, each member of the cycle prefers his successor to his predecessor. Thus we must have

$$x_{k+1} > X_{k+1,k+l}\,, \quad x_{k+2} > X_{k+2,k+1}, \ldots, x_{k+l-1} > X_{k+l-1,k+l-2}\,. \tag{4}$$

Since $X_{\alpha\beta}$ are all independent and [0, 1]-uniform,

$$P((4) \mid x, y) = x_{k+1} \cdot \prod_{\text{even } j=k+2}^{k+l-1} x_j\,. \tag{5}$$

All other conditions relate to the pairs $\{X_{ij}, X_{ji}\}$, $\{i, j\} \not\in D$.

(1) $i = n$, $j \ne 1$. The member 1 is the best choice of the member n among those who prefer n to their predecessors in Π. Hence

$$X_{nj} > y_n \quad \text{or} \quad X_{jn} > x_j\,.$$

Notice that $x_j \ge y_j$ unless j is a proposer. In that case $X_{nj} < y_n$ would imply y_j $(= X_{j,j+1}) < X_{jn}$, since otherwise j would have proposed to n. Therefore, when $y_j > x_j$, we have

$$X_{nj} > y_n \quad \text{or} \quad X_{jn} > y_j\,.$$

So, introducing $x_n = y_n$, for all j

$$X_{nj} > x_n \vee y_n \quad \text{or} \quad X_{jn} > x_j \vee y_j \tag{6}$$

where $a \vee b = \max(a, b)$, and

$$P((6) \mid x, y) = 1 - (x_n \vee y_n)(x_j \vee y_j)\,. \tag{7}$$

(2) $i, j \in \{1, \ldots, n-1\}$, $\{i, j\} \not\subseteq D$.

(2′) $j \not\subseteq \{1, \ldots, k+l\}$, so that $x_j = y_j$. Since Π is stable,

$$X_{ij} > x_i \quad \text{or} \quad X_{ji} > x_j\,.$$

Also, as in (1), if $x_i < y_i$, then

$$X_{ij} > y_i \quad \text{or} \quad X_{ji} > x_j\,,$$

so always

$$X_{ij} > x_i \vee y_i \quad \text{or} \quad X_{ji} > x_j \vee y_j \tag{8}$$

and

$$P((8) \mid x, y) = 1 - (x_i \vee y_i)(x_j \vee y_j)\,.$$

(2″) $i, j \in \{1, \ldots, k+l\}$, $i < j$. If i, j are both even, then $x_i \le y_i$, $x_j \le y_j$ and

$$X_{ij} > y_i \quad \text{or} \quad X_{ji} > y_j\,,$$

since Π' is stable. If i, j are both odd, then $x_i > y_i$, $x_j > y_j$, and

$$X_{ij} > x_i \quad \text{or} \quad X_{ji} > x_j\,,$$

since Π is stable. Suppose i is even and j is odd, so that $x_i \le y_i$, $x_j > y_j$. Using stability of Π if $x_i = y_i$, or arguing as in (1), (2′) if $x_i < y_i$ (remember that $i < j$), we get

$$X_{ij} > y_i \quad \text{or} \quad X_{ji} > x_j\,.$$

Finally, suppose i is odd and j is even, so that $x_i > y_i$, $x_j \le y_j$. Let $x_j < y_j$, i.e., let j be a proposer. Then using condition (ii) of Theorem 3.1 for the first time, we obtain that $X_{ji} < y_j \Rightarrow X_{ij} > x_i$. Thus we must have

$$X_{ij} > x_i \quad \text{or} \quad X_{ji} > y_j\,.$$

So, in all four cases,

$$X_{ij} > x_i \vee y_i \quad \text{or} \quad X_{ji} > x_j \vee y_j \tag{9}$$

and

$$P((9)\,|\,x, y) = 1 - (x_i \vee y_i)(x_j \vee y_j)\,.$$

Since the events (4), (6), (8), and (9) are conditionally independent, we arrive at

$$\begin{aligned} P_n(k, l\,|\,x, y) &= x_{k+1} \cdot \prod_{\text{even } j=k+2}^{k+l-1} x_j \cdot \prod_{\{\alpha,\beta\}\not\in D} [1 - (x_\alpha \vee y_\alpha)(x_\beta \vee y_\beta)] \\ &= z_{k+1} \cdot \prod_{\text{even } j=k+2}^{k+l-1} z_j \cdot \prod_{\{\alpha,\beta\}\not\in D} (1 - z_\alpha z_\beta)\,, \end{aligned} \tag{10}$$

where $z_\alpha = x_\alpha \vee y_\alpha$.

To finish the proof of the lemma, introduce also $\zeta_\alpha = x_\alpha \wedge y_\alpha$ [$=\min(x_\alpha, y_\alpha)$]. The variables z_α, ζ_α satisfy the following conditions [see (2), (3)]:

(a) $z_\alpha = \zeta_\alpha$ for $\alpha \in \{k+l+1, \ldots, n-1, n\}$, and for even α in $\{k+1, \ldots, k+l\}$;
(b) $0 < \zeta_\alpha < z_\alpha < 1$ for odd α in $\{1, 2, \ldots, k+l\}$ and even α in $\{1, \ldots, k\}$.

Denote by A the set of αs in (b). Using Fubini's Theorem and switching to $\{z_\alpha, \zeta_\alpha\}$, we have

$$\begin{aligned} P_n(k, l) &= \int P_n(k, l\,|\,x, y)\, dx\, dy \\ &= \int_{z\in\mathscr{C}} \left\{ z_{k+1} \cdot \prod_{\text{even } j=k+2}^{k+l-1} z_j \cdot \sum_{\{\alpha,\beta\}\not\in D} (1 - z_\alpha z_\beta) \left(\int_{\zeta_\alpha \le z_\alpha} \prod_{\alpha\in A} d\zeta_\alpha \right) \right\} dz \\ &= \int_{z\in\mathscr{C}} \left\{ z_{k+1} \cdot \prod_{j=1}^{k+l} z_j \cdot \prod_{\{\alpha,\beta\}\not\in D} (1 - z_\alpha z_\beta) \right\} dz\,. \end{aligned}$$

The lemma is proven. ∎

Note: The formula for $P_n(k, l)$ and other related formulae in [8], [9], [10], and [11], are all close relatives of a formula for the probability that a given matching (marriage) in a bipartite model is stable, obtained earlier by Knuth [7]. (In fact, much of what was done in [8], [9], [10], and [11] was inspired by that remarkable formula.) Knuth's method, however, was quite different. It relied on the inclusion–exclusion principle applied directly to the random permutations P_i, and on two penetrating observations — first, that each summand in the corresponding formula can be thought of as the value of a certain multidimensional integral; second, that changing the order of summation and integration leads to a single integral of a product-type integrand. Our approach is more pedestrian, but it has worked in a number of situations, like the lemma, where applicability of the inclusion-exclusion method is quite problematic.

Our next step is to use the lemma to obtain asymptotic estimates of $P_n(k, l)$ for (relatively) small and large values of k and l. The method will parallel those used in [8], [9], [10], and [11]. In particular, we will use the following inequality [9].

Lemma 4.2. *For $z \in \mathscr{C}$, define $s = \sum_{i=1}^{n} z_i$, and $v = \{v_i = z_i/s; 1 \le i \le n\}$, so that $\sum_{i=1}^{n} v_i = 1$. Let $L^{(n)} = \{L_i^{(n)} : 1 \le i \le n\}$ be the set of lengths of the consecutive subintervals of $[0, 1]$ obtained by selecting, independently and uniformly at random, $(n-1)$ points in $[0, 1]$. Then, for $f(s)$, $g(v) \ge 0$,*

$$\int_{\mathscr{C}} f(s)g(v)\,dz = \int_0^n f(s)\,\frac{s^{n-1}}{(n-1)!}\,E(g(L^{(n)})\chi(\max_i L_i^{(n)} \le s^{-1}))\,ds$$

$$\le E(g(L^{(n)})) \int_0^n f(s)\,\frac{s^{n-1}}{(n-1)!}\,ds\,,$$

with equality if $sv \in \mathscr{C}$ whenever $f(s) > 0$ and $g(v) > 0$.

This lemma is very helpful since the random partitions are well understood. Here is a particularly useful property:

$$L^{(n)} \overset{\mathscr{D}}{=} \left\{ \frac{w_i}{\sum_{j=1}^{n} w_j}\ ;\ 1 \le i \le n \right\}, \tag{11}$$

where $w_1, \ldots, w_n$ are i.i.d. exponential, exp(1) say (see Breiman [1] and Durrett [2], for instance). For example, we will need to know that $\sum_{j=1}^{n} (L_j^{(n)})^2$ is likely to be of order n^{-1}. Well, using $Ew_1 = 1$, $Ew_1^2 = 2$ and the exponential bounds for the distribution tails of sums of i.i.d. random variables, we immediately get from (11): for every $\epsilon > 0$, there are $n(\epsilon)$, $c(\epsilon)$ such that for $n \ge n(\epsilon)$

$$P\Big(2(1-\epsilon)/n \le \sum_{j=1}^{n} (L_j^{(n)})^2 \le 2(1+\epsilon)/n\Big) \ge 1 - \exp(-c(\epsilon)n)\,. \tag{12}$$

First, we obtain a bound for $P_n(k, l)$ that is not very sharp, but works for all values of k, l. The proof is close to that of Lemma 3.2 in [8], but we include it in order to make the presentation more self-contained.

Lemma 4.3. *Uniformly over k, l $(k + l \le n - 1)$,*

$$P_n(k, l) = O\left(\frac{(n+k+l-1)!!}{(n+k+l)!}\right), \tag{13}$$

and consequently

$$N_n(k, l)P_n(k, l) = O(n^{-1}\exp(-(k+l)^2/4n)\,. \tag{14}$$

Proof.

(a) Turn to (1). Using $1-\zeta \le e^{-\zeta}$ and $ab \le (a^2+b^2)/2$, we bound

$$\begin{aligned}
\prod_{\{\alpha,\beta\}\not\subseteq D} (1 - z_\alpha z_\beta) &\le \exp\Big(-\sum_{\{\alpha,\beta\}\not\subseteq D} z_\alpha z_\beta\Big) \\
&= \exp\Bigg(-\frac{s^2}{2} + \frac{1}{2}\sum_{\alpha=1}^{n} z_\alpha^2 + z_n z_1 + \sum_{\alpha=1}^{k+l-1} z_\alpha z_{\alpha+1} \\
&\qquad + z_{k+l} z_{k+1} + \sum_{\substack{\alpha=k+l+1\\ \alpha \text{ even}}}^{n-2} z_\alpha z_{\alpha+1}\Bigg) \\
&\le c\exp\Big(-\frac{s^2}{2} + \frac{3}{2}\sum_{\alpha=1}^{n} z_\alpha^2\Big)\Big(s \overset{df}{=} \sum_{\alpha=1}^{n} z_\alpha\Big) \qquad (15)
\end{aligned}$$

($\max_\alpha |z_\alpha| \le 1$). So $\{z \in \mathscr{C} : s \ge n^{1/2}\log n\}$ contributes to $P_n(k,l)$ a quantity of order $O(\exp(-\frac{1}{3}n\log^2 n))$, that can be neglected because $\sum_{k,l} N_n(k,l) = O(nn!)$. Thus we concentrate on $\mathscr{C}_1 = \{z \in \mathscr{C} : s \le n^{1/2}\log n\}$. On $\mathscr{C}_1$,

$$\prod_{\{\alpha,\beta\}\not\subseteq D} (1 - z_\alpha z_\beta) \le \exp\Big(-\frac{s^2}{2} + O(n^{1/2}\log n)\Big). \qquad (16)$$

We partition $\mathscr{C}_1$ into $\mathscr{C}_1'$ (where $s^{-2}\sum_{i=1}^{n} z_i^2 \ge 9/n$) and $\mathscr{C}_1'' = \mathscr{C}_1 - \mathscr{C}_1'$. Using Lemmas 4.1, 4.2, and (16), we bound

$$\begin{aligned}
\int_{\mathscr{C}_1'} &\le \exp(O(n^{1/2}\log n)) \int_0^\infty e^{-s^2/2}\, \frac{s^{n+k+l}}{(n-1)!}\, ds \\
&\quad \cdot E\Big(L_{k+1}^{(n)} \prod_{i=1}^{k+l} L_i^{(n)};\ \sum_{j=1}^{n} (L_j^{(n)})^2 \ge \frac{9}{n}\Big). \qquad (17)
\end{aligned}$$

The integral on the right equals $(n+k+l-1)!!/(n-1)!$. Let us bound the expectation—call it $E_n(k,l)$. The joint density $f(y_1,\dots,y_{n-1})$ of $\{L_j^{(n)} : 1 \le j \le n-1\}$ is $(n-1)!$ if $\sum_{j=1}^{n-1} y_j \le 1$ ($y_j \ge 0$), and zero otherwise. So, denoting $y_n = 1 - \sum_{j=1}^{n-1} y_j$,

$$\begin{aligned}
E_n(k,l) &= (n-1)! \int_{\Sigma_j y_j^2 \ge 9/n} y_{k+1}\Big(\prod_{i=1}^{k+l} y_i\Big) \prod_{j=1}^{n-1} dy_j \\
&= (n-1)! \int_{\Sigma_j y_j^2 \ge 9/n} 2(dy'_{k+1}dy''_{k+1}dy'''_{k+1}) \Bigg(\prod_{\substack{i=1\\ (i\ne k+1)}}^{k+l} dy'_i dy''_i\Bigg) \prod_{j=k+l+1}^{n-1} dy_j,
\end{aligned}$$

($y_{k+1} = y'_{k+1} + y''_{k+1} + y'''_{k+1}$, $y_i = y'_i + y''_i$ for $1 \le i \le k+l$ and $i \ne k+1$). Using $(a+b+c)^2 \le 3\,(a^2+b^2+c^2)$, and renaming the variables, we obtain according to (12):

$$\begin{aligned} E_n(k,l) &\le \frac{2(n-1)!}{(n+k+l)!}(n+k+l)! \int_{\sum_{j=1}^{n+k+l+1} y_j^2 \ge 3/n} \prod_{j=1}^{n+k+l} dy_j \\ &= \frac{2(n-1)!}{(n+k+l)!} P\left(\sum_{j=1}^{n+k+l+1} (L_j^{(n+k+l+1)})^2 \ge \frac{3}{n}\right) \\ &= O\left(e^{-cn}\frac{(n-1)!}{(n+k+l)!}\right), \quad c>0. \end{aligned}$$

So, by (17),

$$\int_{\mathscr{C}_1'} = O\left(e^{-c'n}\frac{(n+k+l-1)!!}{(n+k+l)!}\right). \tag{18}$$

Furthermore,

$$\begin{aligned} \int_{C_1''} &\le \int_{\mathscr{C}} \exp\left[-\frac{s^2}{2}\left(1-\frac{27}{n}\right)\right]\left(z_{k+1}\prod_{j=1}^{k+l} z_j\right) dz \\ &\le \int_0^\infty \exp\left[-\frac{s^2}{2}\left(1-\frac{27}{n}\right)\right]\frac{s^{n+k+l}}{(n+k+l)!}\, ds \\ &= \left(1-\frac{27}{n}\right)^{-(n+k+l+1)/2}\cdot\frac{(n+k+l-1)!!}{(n+k+l)!}. \end{aligned}$$

Since the first factor on the right is bounded, using this bound with (18), we obtain (13).

(b) Set $k+l=r$. Then, using (13) and $N_n(k,l)=(n-1)!/(n-k-l-1)!!$,

$$\begin{aligned} N_n(k,l)P_n(k,l) &= O\left(\frac{(n-1)!}{(n-r-1)!!}\,\frac{(n+r-1)!!}{(n+r)!}\right) \\ &= O\left(n^{-1}\prod_{j=1}^{r}\left(1+\frac{r}{n}\right)^{-1}\right). \end{aligned}$$

The product is bounded by

$$\begin{aligned} \exp\left(n\int_0^{r/n}\log\frac{1}{1+x}\,dx\right) &\le \exp\left[-n\frac{1}{2(1+r/n)}\left(\frac{r}{n}\right)^2\right] \\ &\le \exp\left(-\frac{r^2}{4n}\right). \end{aligned}$$

The lemma is proven. ■

The next estimate is a direct consequence of (14).

Corollary 4.1. *Let* $r_0=\lceil n^{1/2}\log n\rceil$. *Then*

$$\sum_{k+l\ge r_0} N_n(k,l)P_n(k,l) = O(\exp(-\log^2 n/4)).$$

So it remains to consider $k+l\le r_0$. The proof shows that for $\mathscr{C}_2=\{z\in \mathscr{C}\,;\, s\le n^{1/2}\log n,\ s^{-2}\sum_{i=1}^{n} z_i^2\le 9/n\}$

$$\int_{\mathscr{C}_2^c} = O\left(n^{-K}\,\frac{(n+k+l-1)!!}{(n+k+l)!}\right),\quad \forall K>0\,. \tag{19}$$

On $\mathscr{C}_2$, the bound (15) simplifies to

$$\prod_{\{\alpha,\beta\}\not\subseteq D}(1-z_\alpha z_\beta)\le c\exp\left[-\frac{s^2}{2}\left(1-\frac{27}{n}\right)\right]. \tag{20}$$

Now, like $\sum_{j=1}^{n}(L_j^{(n)})^2$, $\sum_{\alpha\text{ even}} L_\alpha^{(n)}L_{\alpha+1}^{(n)}$, and $\sum_\alpha (L_\alpha^{(n)})^4$ are relatively close (with exponentially high probability) to their expectations, which are asymptotically $(2n)^{-1}$ and $24n^{-3}$, respectively. Besides,

$$\begin{aligned}P\left(\max_\alpha L_\alpha^{(n)}\le(1+\rho)\,\frac{\log n}{n}\right)&\ge 1-nP\left(L_1^{(n)}\ge(1+\rho)\,\frac{\log n}{n}\right)\\&=1-n\left(1-(1+\rho)\,\frac{\log n}{n}\right)^{n-1}\\&\ge 1-O(n^{-\rho})\,,\quad \forall\rho>0\,.\end{aligned}$$

So, analogously to the proof of Lemma 4.3, we can show [with the help of (20)] that

$$\int_{\mathscr{C}_2}=\int_{\mathscr{C}_3}+O\left(n^{-\rho}\,\frac{(n+k+l-1)!!}{(n+k+l)!}\right). \tag{21}$$

Here $\mathscr{C}_3$ is a subset of $\mathscr{C}_2$ defined by the additional constraints

$$(1-\epsilon)2/n\le s^{-2}\sum_\alpha z_\alpha^2\le(1+\epsilon)2/n\,,$$

$$(1-\epsilon)(2n)^{-1}\le s^{-2}\sum_{\alpha\text{ even}} z_\alpha z_{\alpha+1}\le(1+\epsilon)(2n)^{-1}\,,$$

$$(1-\epsilon)24/n^3\le s^{-4}\sum_\alpha z_\alpha^4\le(1+\epsilon)24/n^3\,,$$

$$s^{-1}\max_\alpha z_\alpha\le(1+\rho)\log n/n\,.$$

($\epsilon>0$, $\rho>0$ are arbitrary.) For $k+l\le r_0$, on the set $\mathscr{C}_3$ the bound (15) is easily sharpened as follows:

$$\begin{aligned}\prod_{\{\alpha,\beta\}\not\subseteq D}(1-z_\alpha z_\beta)&=\exp\left(-\sum_{\{\alpha,\beta\}\not\subseteq D}[z_\alpha z_\beta+\tfrac12 z_\alpha^2z_\beta^2+O(z_\alpha^3z_\beta^3)]\right)\\&=\exp\left(-\frac{s^2}{2}\left\{1-\frac{3}{n}\,[1+O(\epsilon)]\right\}-\frac{s^4}{n^2}\,[1+O(\epsilon)]+O\left(\frac{\log^5 n}{n^{1/2}}\right)\right).\end{aligned} \tag{22}$$

But, for fixed a and $b > 0$,

$$\int_{\mathscr{C}} \exp\left[-\frac{s^2}{2}\left(1-\frac{a}{n}\right)-\frac{s^4}{n^2}b\right] z_{k+1}\left(\prod_{i=1}^{k+l} z_i\right) dz$$

$$\leq 2\int_0^\infty \exp\left[-\frac{s^2}{2}\left(1-\frac{a}{n}\right)-\frac{s^4}{n^2}b\right]\frac{s^{n+k+l}}{(n+k+l)!}\,ds$$

$$\sim 2e^{a/2}\int_0^\infty \exp\left(-\frac{s^2}{2}-\frac{s^4}{n^2}b_n\right)\frac{s^{n+k+l}}{(n+k+l)!}\,ds[b_n = b(1-a/n)^{-2}] \tag{23}$$

$$\sim 2e^{a/2-b}\int_0^\infty e^{-s^2/2}\frac{s^{n+k+l}}{(n+k+l)!}\,ds$$

$$= 2e^{a/2-b}\,\frac{(n+k+l-1)!!}{(n+k+l)!}\,. \tag{24}$$

[Since $n+k+l \sim n$, the integrand in (23) has a sharp maximum at $\bar{s} \sim \sqrt{n}$.]

Collecting the estimates (19), (21), (22), and (24), and letting $\epsilon \downarrow 0$, we prove the following result.

Lemma 4.4. *Uniformly over $k+l \leq r_0$,*

$$P_n(k,l) \lesssim 2e^{1/2}\,\frac{(n+k+l-1)!!}{(n+k+l)!}\,.$$

Using Lemma 4.2 (the case of equality), we can show that $P_n(k,l)$ is equivalent to the upper bound (cf. [9] and [10]). We omit the details.

Consequently, for $r = k+l \leq r_0$ $(=n^{1/2}\log n)$,

$$N_n(k,l)P_n(k,l) \sim 2e^{1/2}\,\frac{(n-1)!(n+r-1)!!}{(n+r)!(n-r-1)!!}$$

$$= 2e^{1/2}n^{-1}\,\frac{\prod_{j=1}^{(r-1)/2}\left(1-\left(\frac{2j}{n}\right)^2\right)}{\prod_{j=1}^{r}\left(1+\frac{j}{n}\right)}$$

$$= 2e^{1/2}n^{-1}\exp\left[-\frac{r^2}{2n}+O\left(\frac{r^3}{n^2}\right)\right].$$

Now k is even, l is odd, so

$$\sum_{k+l\leq r_0} N_n(k,l)P_n(k,l) \sim \frac{e^{1/2}}{2}\int_0^\infty\int_0^\infty \exp\left(-\frac{(x+y)^2}{2}\right)dx\,dy$$

$$= \frac{e^{1/2}}{2}\int_0^\infty ue^{-u^2/2}\,du$$

$$= \frac{e^{1/2}}{2}\,.$$

TABLE 1. Empirical evidence on solvability probability

Size of instance	% Solvable	Size of instance	% Solvable
100	64.4	1100	36.8
200	53.9	1200	38.5
300	48.5	1300	36.1
400	46.2	1400	36.1
500	46.5	1500	36.8
600	43.1	1600	36.5
700	41.2	1700	35.4
800	40.0	1800	32.4
900	38.8	1900	33.5
1000	38.6	2000	32.6

This completes the proof of Theorem 4.1.

As an immediate corollary of Theorem 4.1 and the remarks preceding it, we have our main result.

Corollary 4.2. $\overline{\lim}_{n\to\infty} P_n \leq e^{1/2}/2$.

5. EMPIRICAL EVIDENCE

So we have established a first upper bound on the limiting behavior of the roommates solvability probability P_n. However, it remains an open problem as to whether P_n is bounded below by some constant greater than 0.

The available empirical evidence here is not conclusive, though it does indicate that the upper bound established here $(0.8244\ldots)$ is not likely to be very tight. Table I shows the percentage that were solvable among 1000 randomly generated roommates instances of each of sizes $100, 200, \ldots, 2000$. The simulations were performed on a Sun4 workstation, size 2000 being the limit imposed by the memory requirements of the program. Although quite striking, this evidence is not really conclusive enough to add support to any strong conjecture as to the ultimate behavior of P_n.

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