# Photons and Quantum Information

*Stephen M. Barnett*

1. A bit about photons

2. Optical polarisation

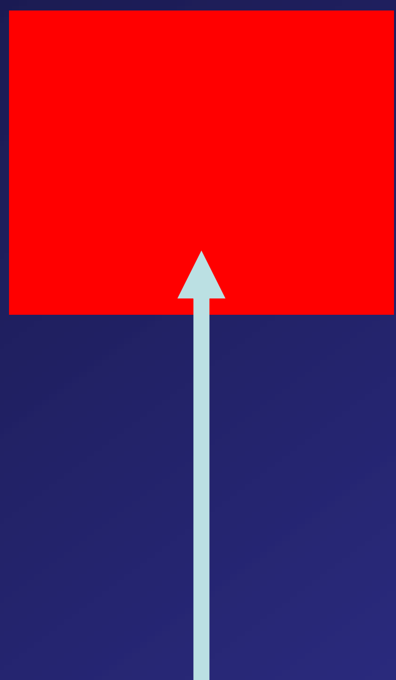
3. Generalised measurements

4. State discrimination
   - Minimum error
   - Unambiguous
   - Maximum confidence

# 1. A bit about photons

Photoelectric effect - Einstein 1905

$$eV = h\nu - W$$



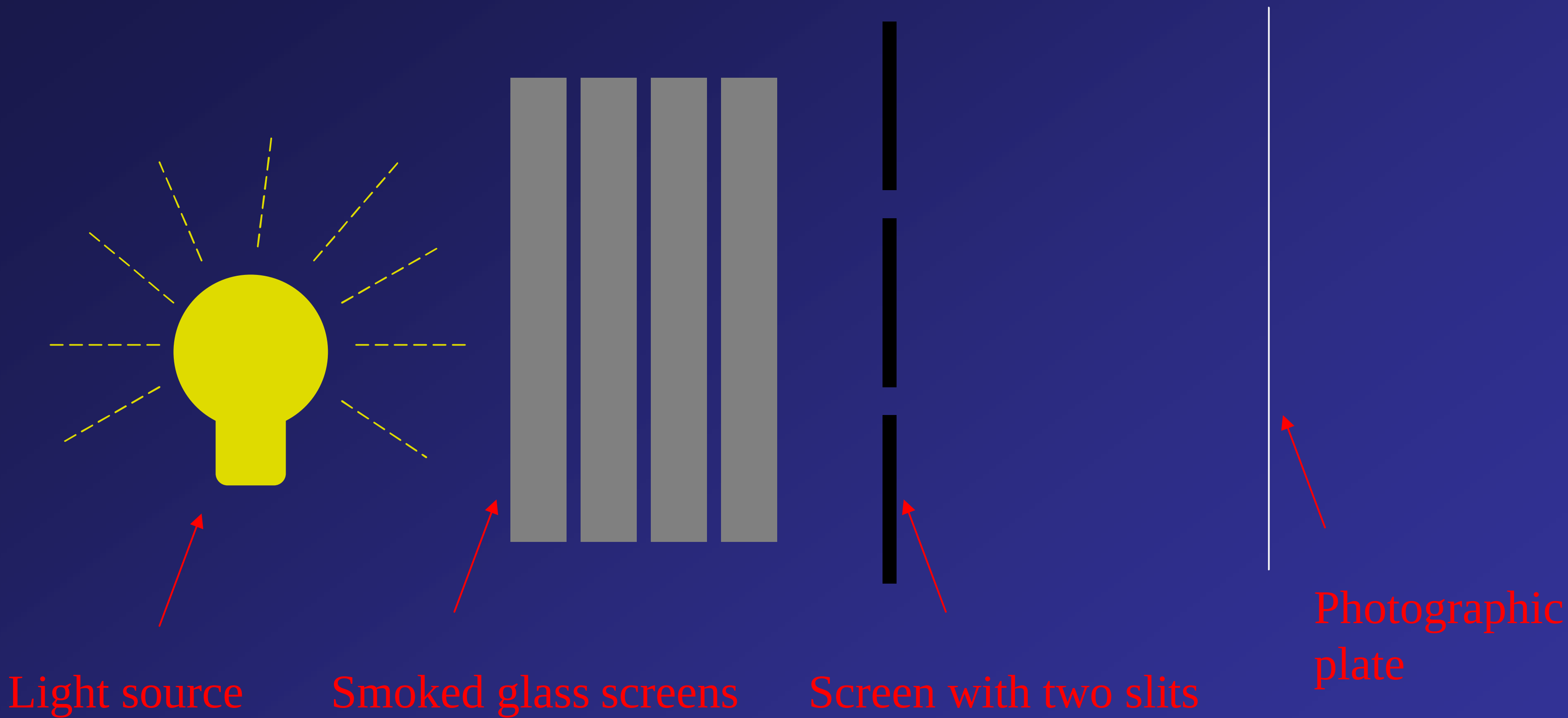
BUT …

Modern interpretation: resonance with the atomic transition frequency.

We can describe the phenomenon quantitatively by a model in which the matter is described quantum mechanically but the light is described classically.

Photons?

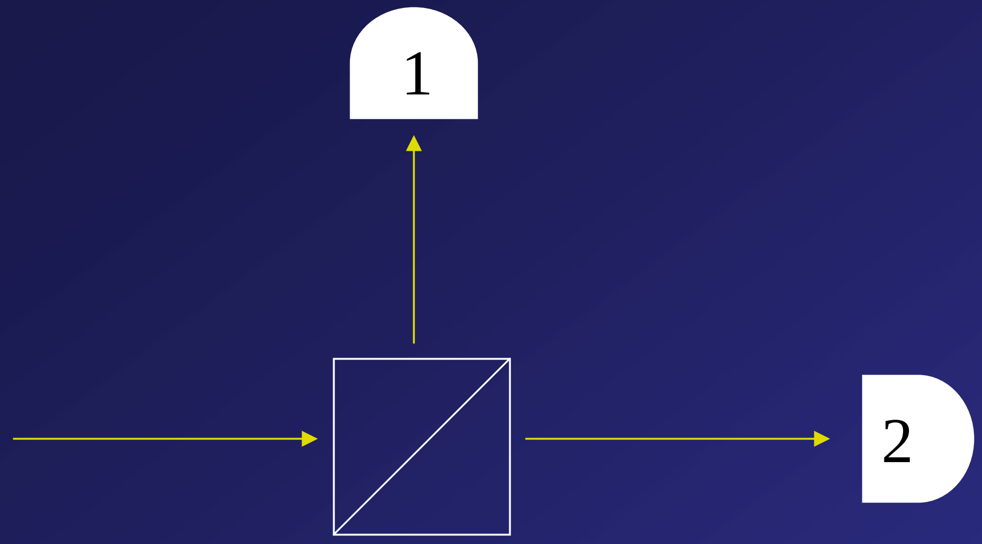
# Single-photon (?) interference - G. I. Taylor 1909




Longest exposure - three months

“According to Sir J. J. Thompson, this sets a limit on the size of the indivisible units.”

Single photons (?) Hanbury-Brown and Twiss




$$P(1,2) \propto \langle RI \times TI \rangle = RT\langle I^2 \rangle$$

$$P(1) \propto R\langle I \rangle \quad P(2) \propto T\langle I \rangle$$

$$g^{(2)}(0) = \frac{\langle I^2 \rangle}{\langle I \rangle^2} \geq 1$$

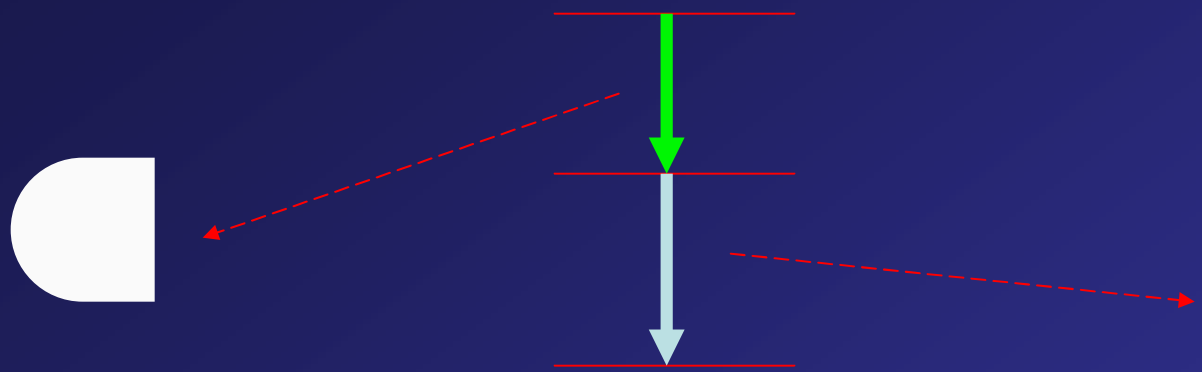
Blackbody light $g^{(2)}(0) = 2$

Laser light $g^{(2)}(0) = 1$

Single photon $g^{(2)}(0) = 0$ !!!

violation of Cauchy-Swartz inequality

# Single photon source - Aspect 1986



Detection of the first photon acts as a herald for the second.

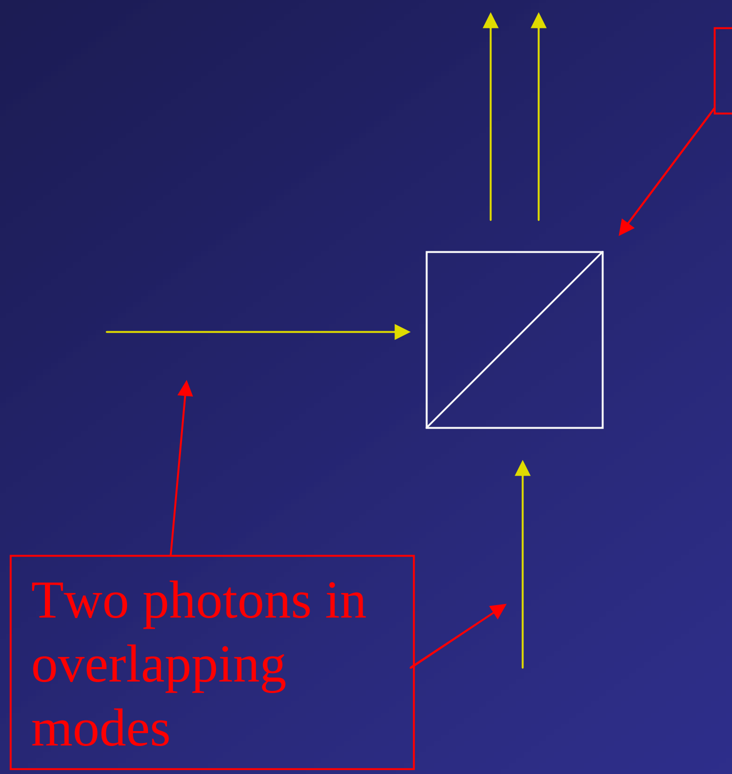
Second photon available for Hanbury-Brown and Twiss measurement or interference measurement.

Found $g^{(2)}(0) \sim 0$ (single photons) and fringe visibility = 98%

# Two-photon interference - Hong, Ou and Mandel 1987

"Each photon then interferes only with itself. Interference between different photons never occurs" Dirac

50/50 beam splitter R=T=1/2

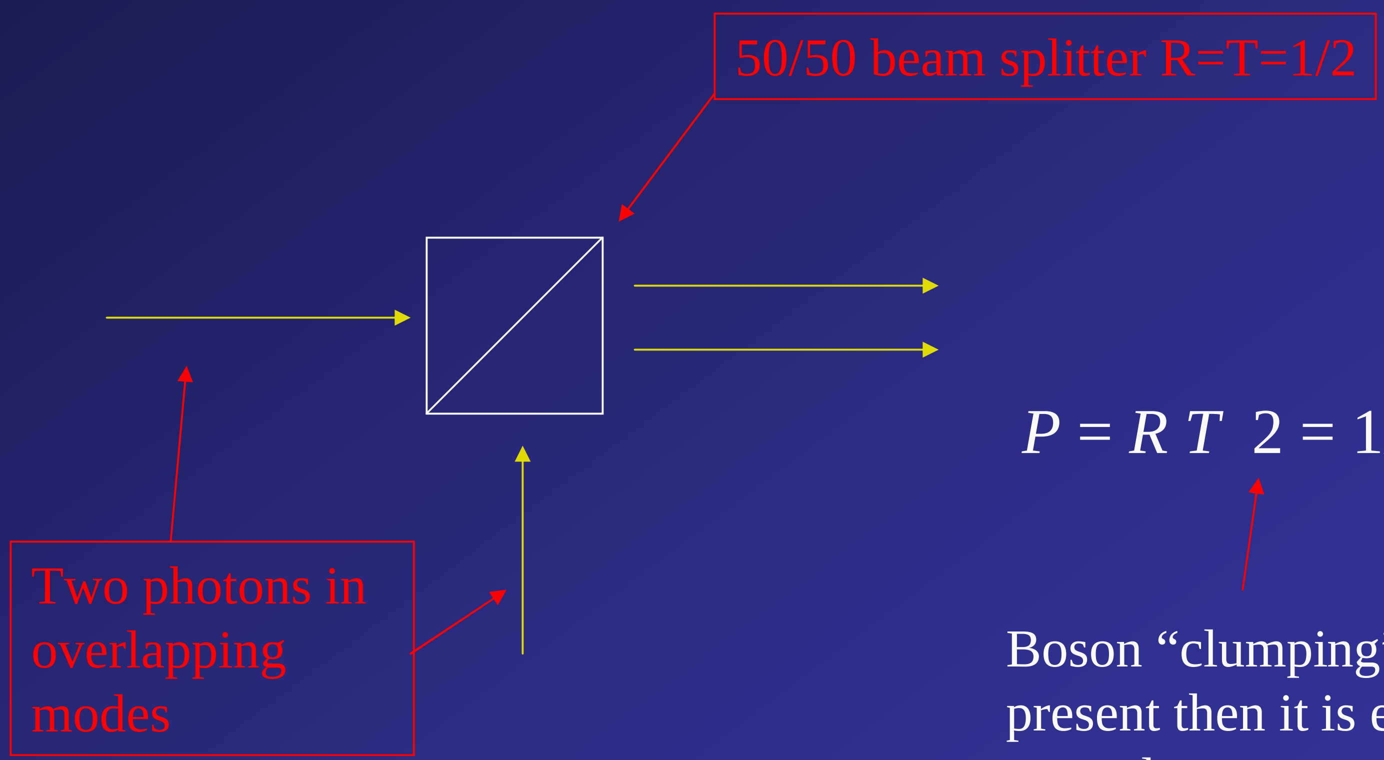



$$P = R\ T\ \ 2 = 1/2$$

Boson "clumping". If one photon is present then it is easier to add a second.

# Two-photon interference - Hong, Ou and Mandel 1987

"Each photon then interferes only with itself.  Interference between different photons never occurs"     Dirac

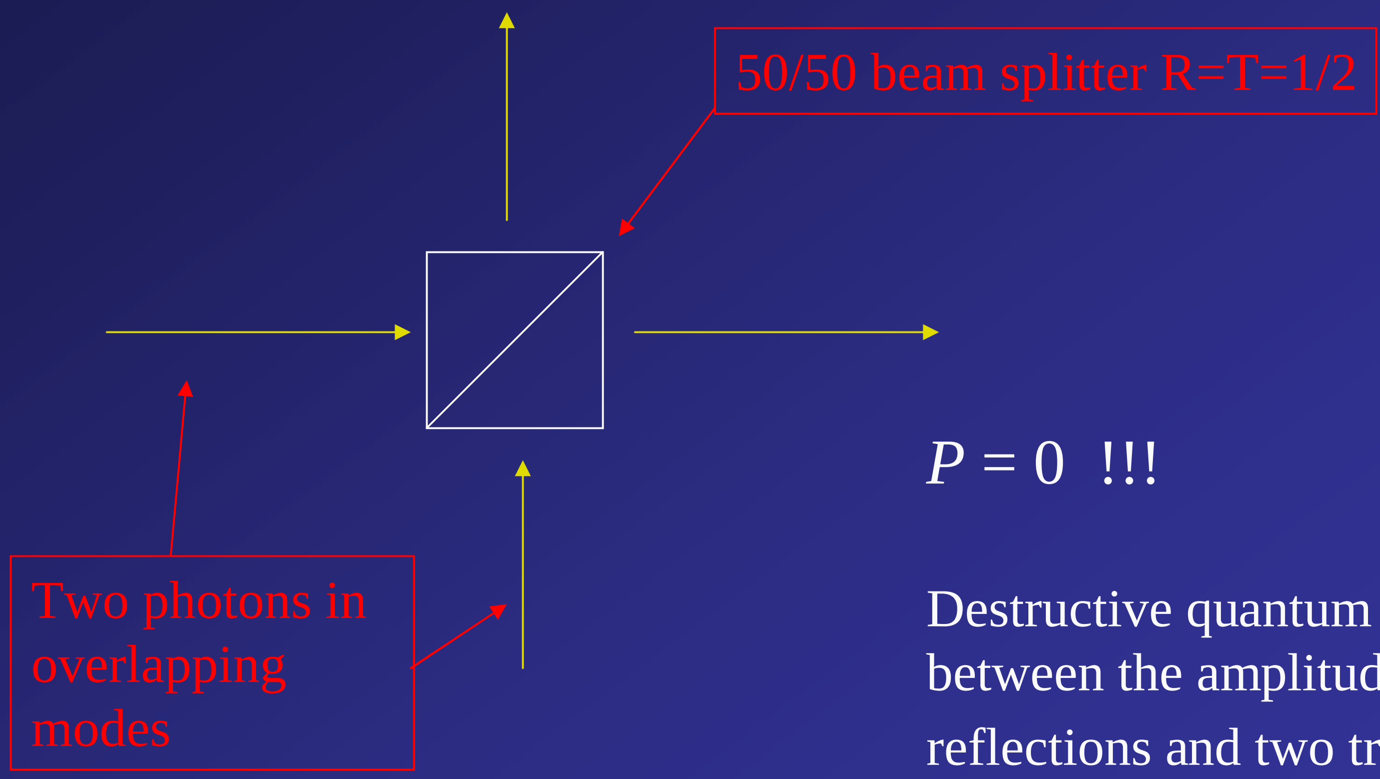



$$P = R \; T \;\; 2 = 1/2$$

Boson "clumping".  If one photon is present then it is easier to add a second.

# Two-photon interference - Hong, Ou and Mandel 1987

"Each photon then interferes only with itself.  Interference between different photons never occurs"     Dirac




$P = 0$  !!!

Destructive quantum interference between the amplitudes for two reflections and two transmissions.

1. A bit about photons

2. Optical polarisation

3. Generalised measurements

4. State discrimination

   Minimum error

   Unambiguous

   Maximum confidence

Maxwell’s equations in an isotropic dielectric medium take the form:

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \frac{\varepsilon}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

**E**, **B** and **k** are mutually orthogonal

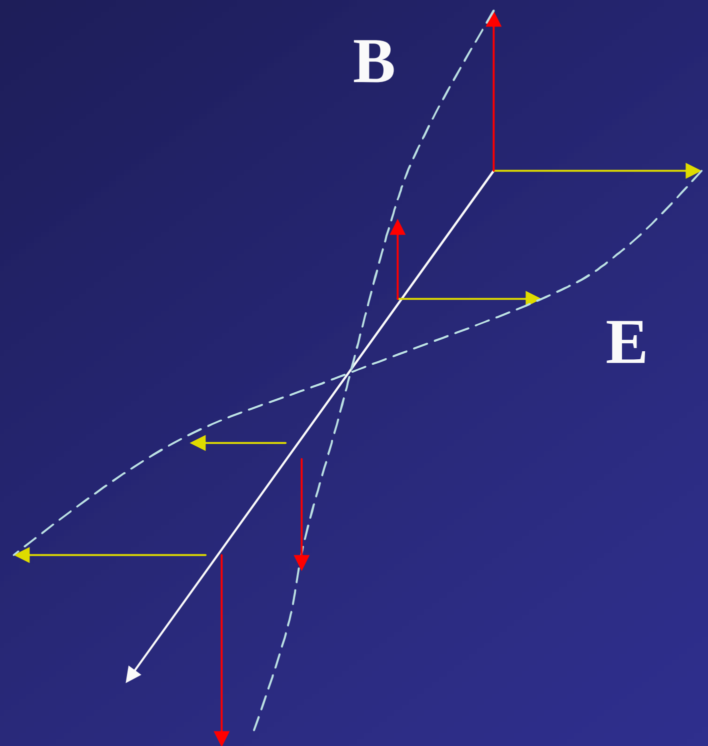
For plane waves (and lab. beams that are not too tightly focussed) this means that the E and B fields are constrained to lie in the plane perpendicular to the direction of propagation.


$$\mathbf{S} = \mu_0^{-1}\mathbf{E} \times \mathbf{B}$$

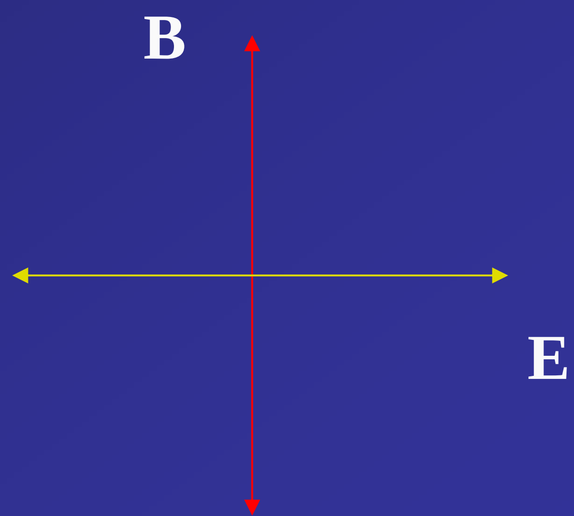
Consider a plane EM wave of the form

$$\mathbf{E} = \mathbf{E}_0 \exp\left[i(kz - \omega t)\right]$$
$$\mathbf{B} = \mathbf{B}_0 \exp\left[i(kz - \omega t)\right]$$

If $\mathbf{E}_0$ and $\mathbf{B}_0$ are constant and real then the wave is said to be linearly polarised.



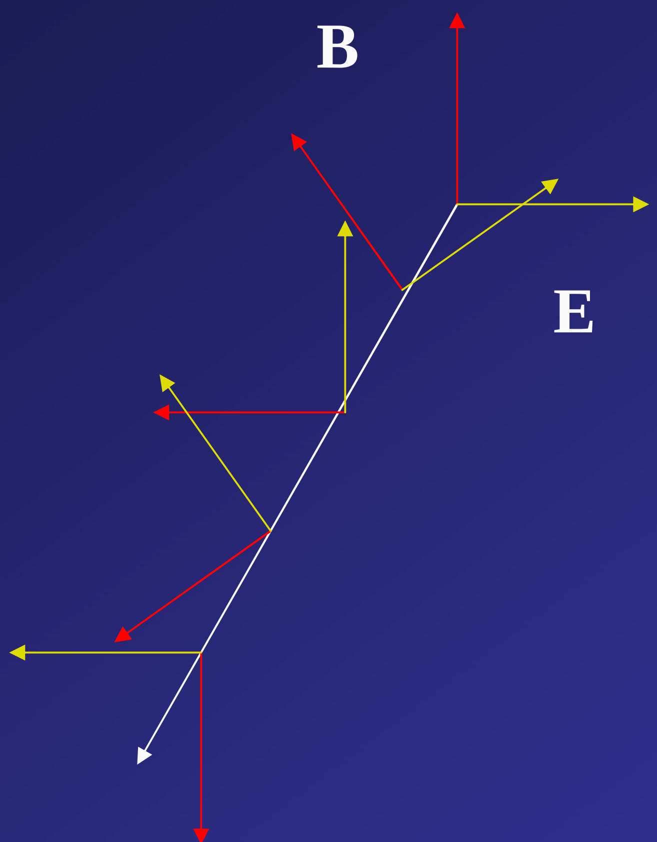

Polarisation is defined by an axis rather than by a direction:



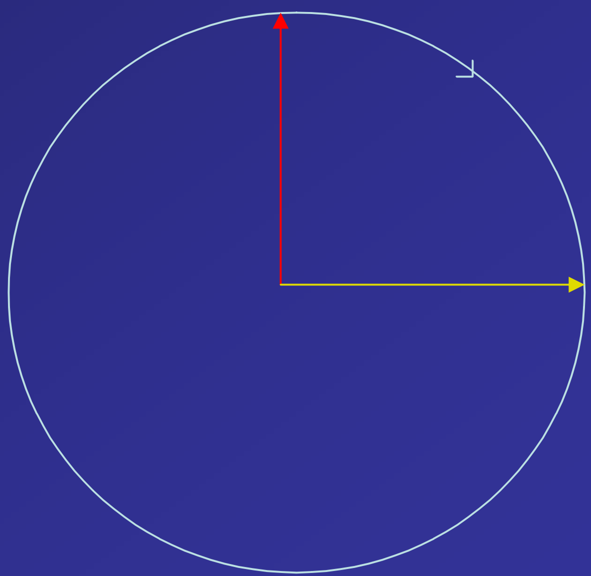
If the electric field for the plane wave can be written in the form

$$\mathbf{E} = E_0(\mathbf{i} \pm i\mathbf{j})\exp\left[i(kz - \omega t)\right]$$

Then the wave is said to be circularly polarised.




For right-circular polarisation, an observer would see the fields rotating clockwise as the light approached.



# The Jones representation

We can write the x and y components of the complex electric field amplitude in the form of a column vector:

$$\begin{bmatrix} E_{0x} \\ E_{0y} \end{bmatrix} = \begin{bmatrix} |E_{0x}| e^{i\phi_x} \\ |E_{0y}| e^{i\phi_y} \end{bmatrix}$$

The size of the total field tells us nothing about the polarisation so we can conveniently normalise the vector:

Horizontal polarisation $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Left circular polarisation $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$

Vertical polarisation $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Right circular polarisation $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$

One advantage of this method is that it allows us to describe the effects of optical elements by matrix multiplication:

Linear polariser
(oriented to horizontal):

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} 0^\circ, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} 90^\circ, \tfrac{1}{2}\begin{bmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{bmatrix} \pm 45^\circ$$

Quarter-wave plate
(fast axis to horizontal):

$$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} 0^\circ, \quad \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} 90^\circ, \quad \tfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & \pm i \\ \pm i & 1 \end{bmatrix} \pm 45^\circ$$

Half-wave plate
(fast axis horizontal or vertical):

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The effect of a sequence of $n$ such elements is:

$$\begin{bmatrix} A \\ B \end{bmatrix} \to \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix} \cdots \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

We refer to two polarisations as orthogonal if

$$\mathbf{E}_2^* \cdot \mathbf{E}_1 = 0$$

This has a simple and suggestive form when expressed in terms of the Jones vectors:

$$\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} \text{ is orthogonal to } \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} \text{ if}$$

$$A_2^* A_1 + B_2^* B_1 = 0$$

$$\Rightarrow \begin{bmatrix} A_2^* & B_2^* \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = 0$$

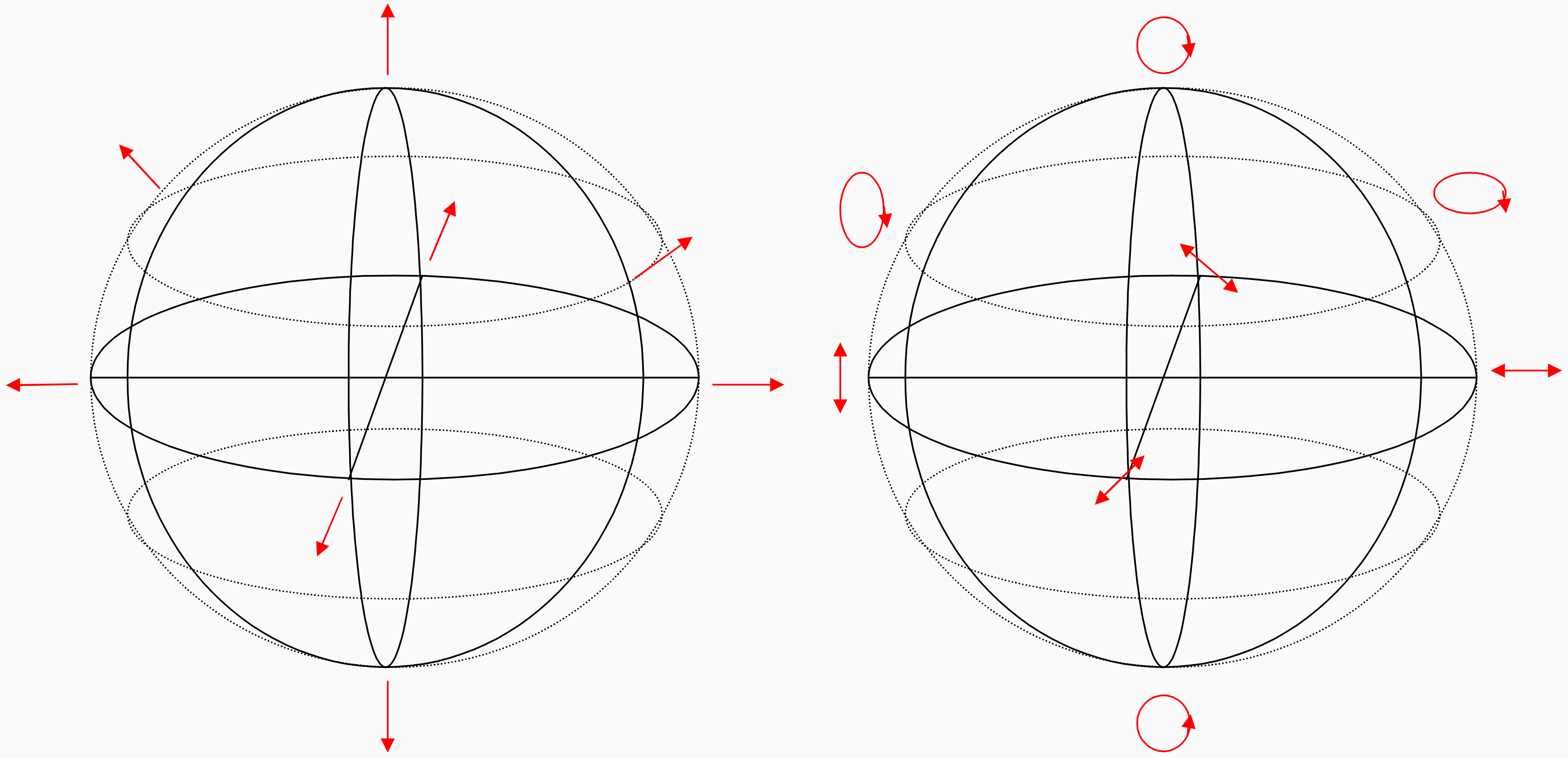
$$\Rightarrow \begin{bmatrix} A_2 \\ B_2 \end{bmatrix}^\dagger \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = 0$$

There is a clear and simple mathematical analogy between the Jones vectors and our description of a qubit.

# Spin and polarisation Qubits

# Poincaré and Bloch Spheres

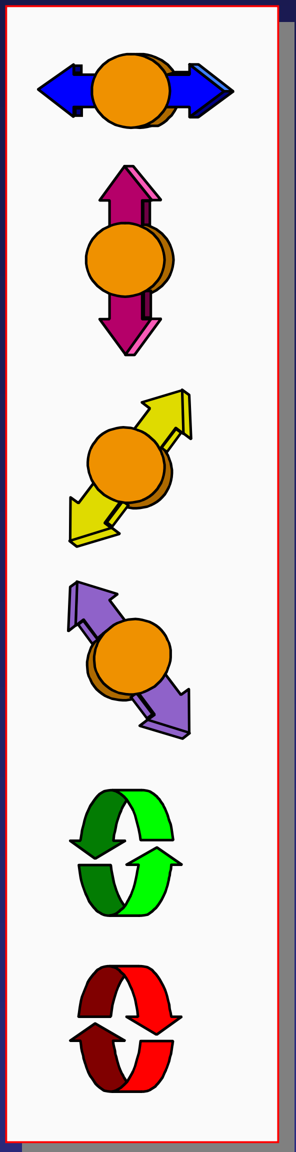
## Two state quantum system



**Bloch Sphere**
Electron spin

**Poincaré Sphere**
Optical polarization

We can realise a qubit as the state of single-photon polarisation



| | |
|---|---|
| Horizontal | $\|0\rangle$ |
| Vertical | $\|1\rangle$ |
| Diagonal up | $\frac{1}{\sqrt{2}}\left(\|0\rangle + \|1\rangle\right)$ |
| Diagonal down | $\frac{1}{\sqrt{2}}\left(\|0\rangle - \|1\rangle\right)$ |
| Left circular | $\frac{1}{\sqrt{2}}\left(\|0\rangle + i\|1\rangle\right)$ |
| Right circular | $\frac{1}{\sqrt{2}}\left(\|0\rangle - i\|1\rangle\right)$ |

1. A bit about photons

2. Optical polarisation

3. Generalised measurements

4. State discrimination

    Minimum error

    Unambiguous

    Maximum confidence

# Probability operator measures

Our generalised formula for measurement probabilities is

$$P(i) = \mathrm{Tr}\left(\hat{\pi}_i \hat{\rho}\right)$$

The set probability operators describing a measurement is called a probability operator measure (POM) or a positive operator-valued measure (POVM).

The probability operators can be defined by the properties that they satisfy:

## Properties of probability operators

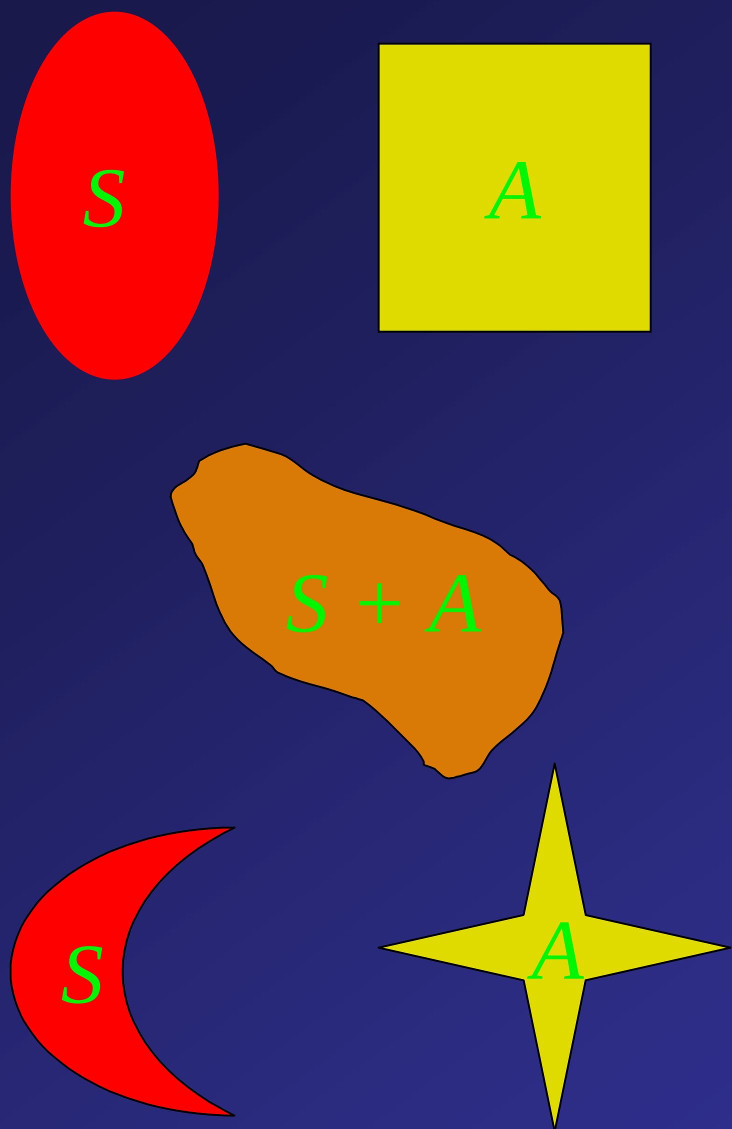
I. They are Hermitian $\hat{\pi}_n^\dagger = \hat{\pi}_n$ Observable

II. They are positive $\langle \psi | \hat{\pi}_n | \psi \rangle \geq 0 \quad \forall |\psi\rangle$ Probabilities

III. They are complete $\sum_n \hat{\pi}_n = \hat{I}$ Probabilities

~~IV. Orthonormal $\hat{\pi}_i \hat{\pi}_j = \delta_{ij} \hat{\pi}_i$~~ ??

# Generalised measurements as comparisons




Prepare an ancillary system in a known state:

$$\left|\psi_S\right\rangle \otimes \left|A\right\rangle$$

Perform a selected unitary transformation to couple the system and ancilla:

$$\hat{U}\left|\psi_S\right\rangle \otimes \left|A\right\rangle$$

Perform a von Neumann measurement on both the system and ancilla:

$$\left|i\right\rangle\rangle = \left|\phi_S^i\right\rangle \otimes \left|A^i\right\rangle$$

The probability for outcome *i* is

$$P(i) = \langle\langle i|\hat{U}|A\rangle|\psi_S\rangle\langle\psi_S|\langle A|\hat{U}^\dagger|i\rangle\rangle$$

$$= \langle\psi_S|\underbrace{\left(\langle A|\hat{U}^\dagger|i\rangle\rangle\langle\langle i|\hat{U}|A\rangle\right)}_{\hat{\pi}_i}|\psi_S\rangle$$

The probability operators $\hat{\pi}_i$ act only on the system state-space.

POM rules:

I. Hermiticity:

$$\left\{\langle A|\hat{U}^\dagger|i\rangle\rangle\langle\langle i|\hat{U}|A\rangle\right\}^\dagger$$

$$= \langle A|\hat{U}^\dagger|i\rangle\rangle\langle\langle i|\hat{U}|A\rangle$$

II. Positivity:

$$\langle\psi|\hat{\pi}_i|\psi\rangle = \left|\langle\langle i|\hat{U}|\psi_S\rangle|A\rangle\right|^2 \geq 0$$

III. Completeness follows from:

$$\sum_i |i\rangle\rangle\langle\langle i| = \hat{I}_{A,S}$$

## Generalised measurements as comparisons

We can rewrite the detection probability as

$$P(i) = \langle A| \otimes \langle \psi_S |\hat{P}_i| \psi_S \rangle \otimes |A\rangle$$

$$\hat{P}_i = \hat{U}^\dagger |i\rangle\rangle\langle\langle i|\hat{U}$$

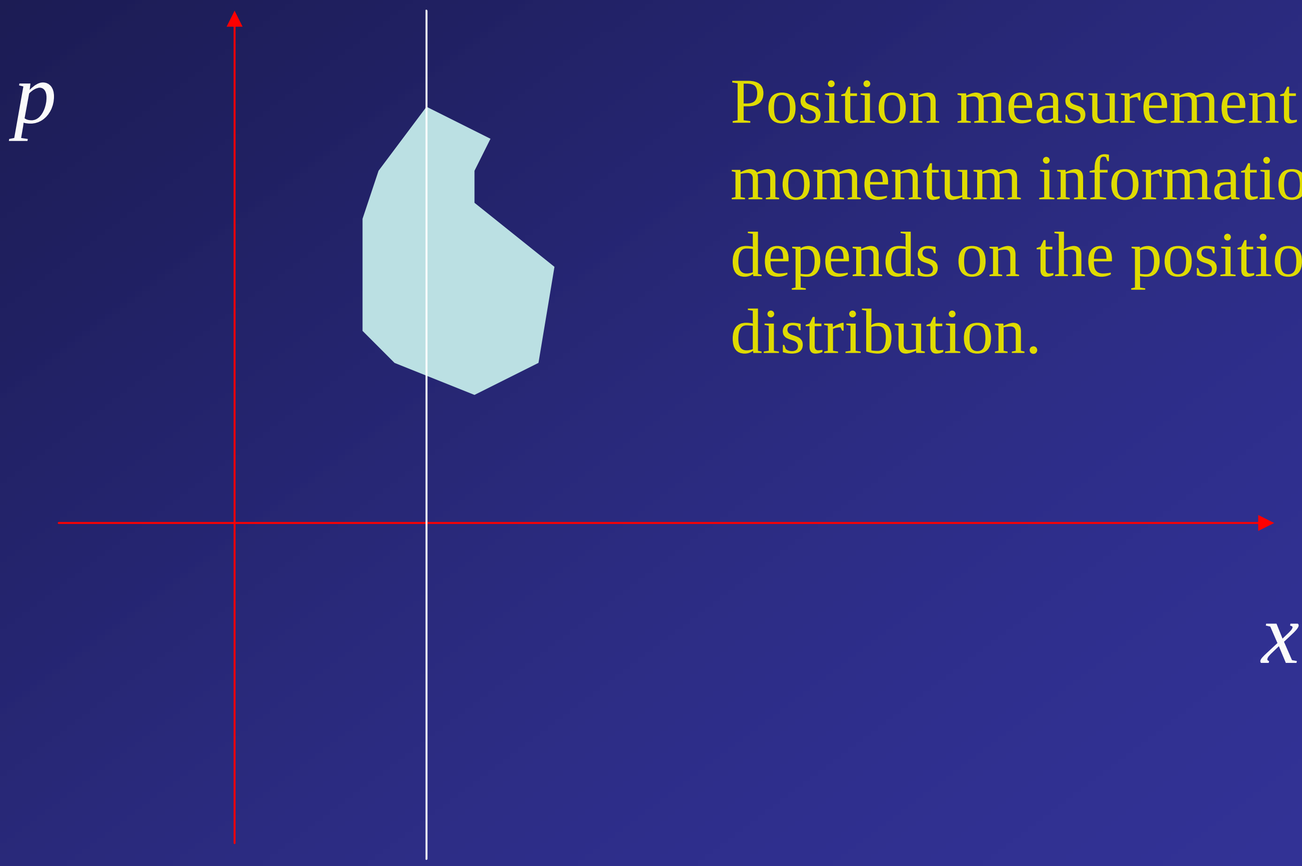
is a projector onto correlated (entangled) states of the system and ancilla. The generalised measurement is a von Neumann measurement in which the system and ancilla are compared.

$$\hat{\pi}_i = \langle A|\hat{P}_i|A\rangle$$

$$\hat{\pi}_n \hat{\pi}_m = \langle A|\hat{P}_n|A\rangle\langle A|\hat{P}_m|A\rangle \neq 0$$

## Simultaneous measurement of position and momentum

The simultaneous *perfect* measurement of $x$ and $p$ would violate complementarity.

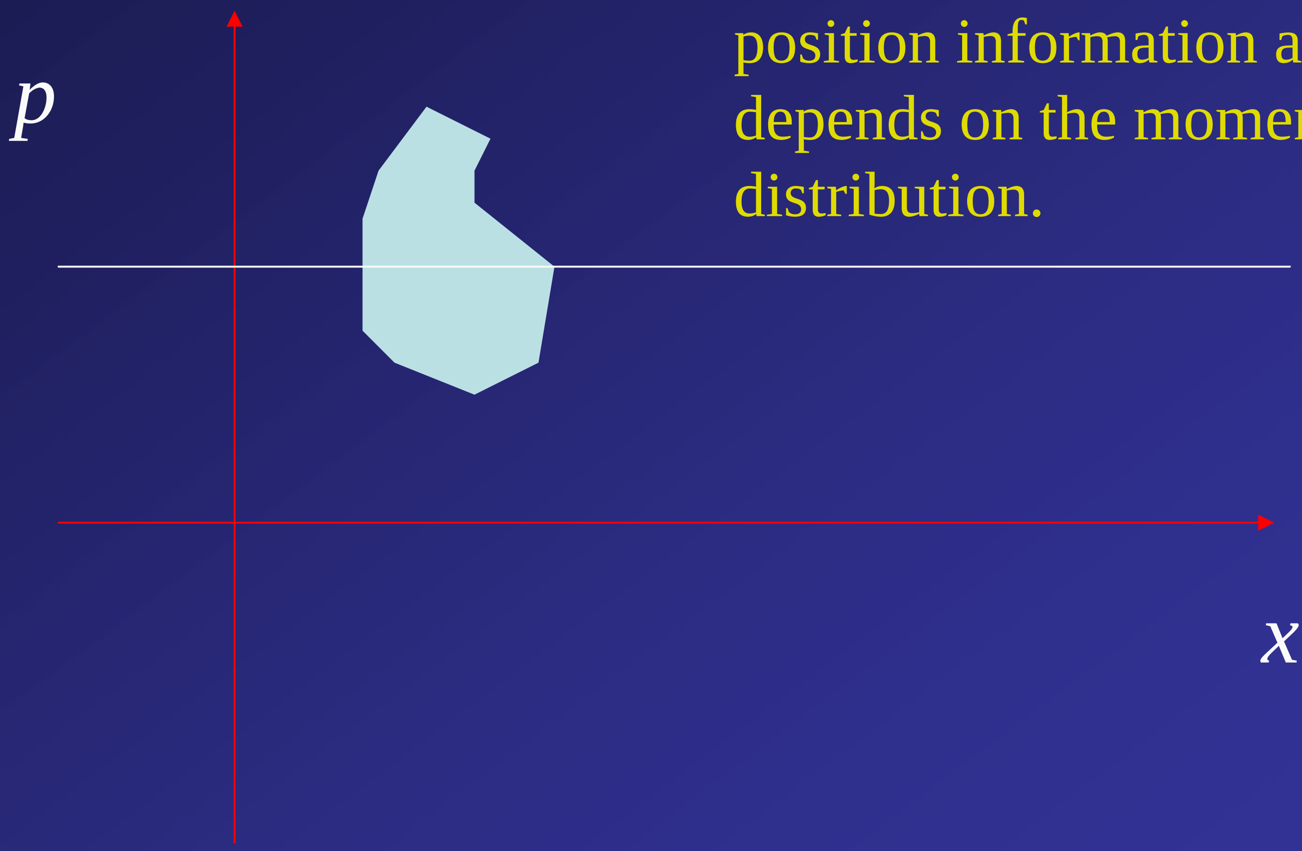



Position measurement gives no momentum information and depends on the position probability distribution.

## Simultaneous measurement of position and momentum

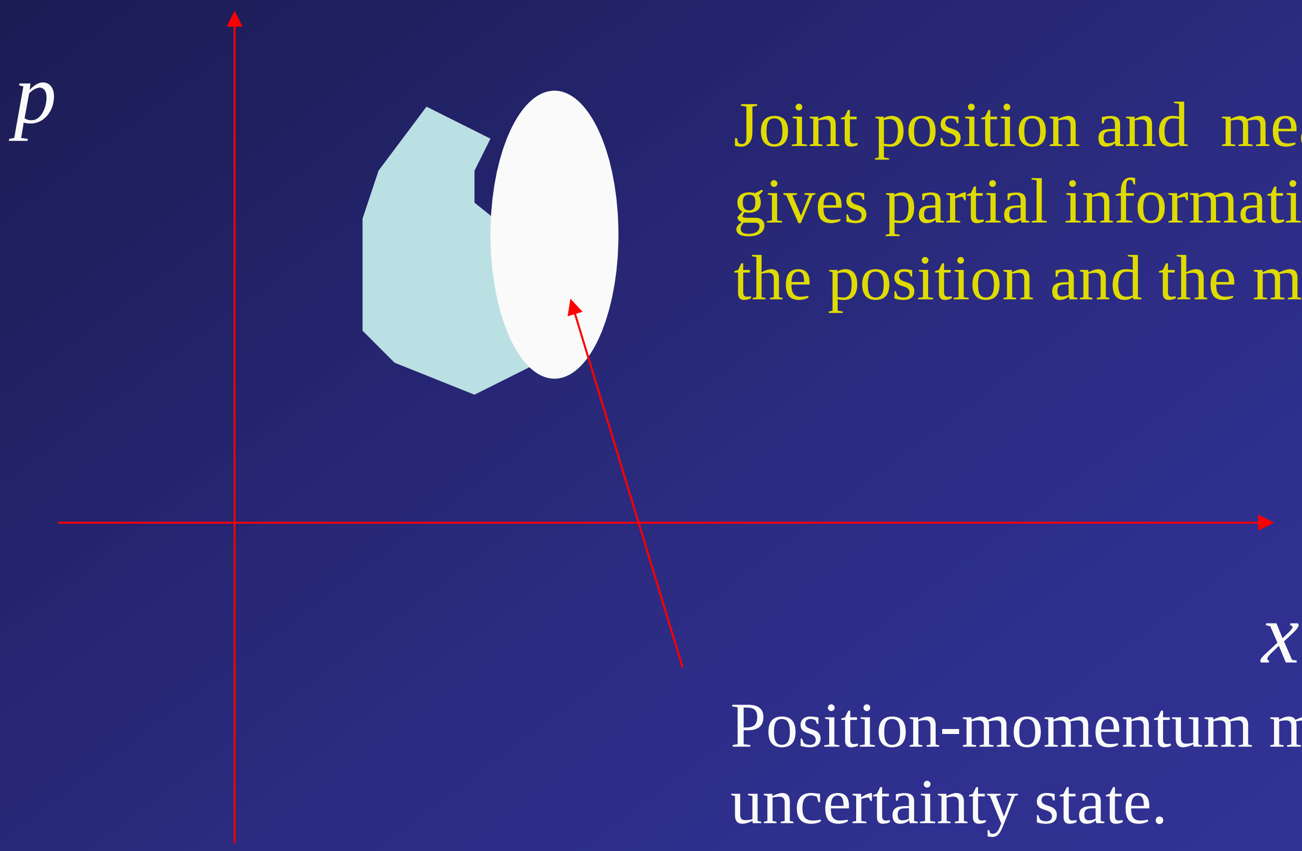
The simultaneous *perfect* measurement of $x$ and $p$ would violate complementarity.

Momentum measurement gives no position information and depends on the momentum probability distribution.



# Simultaneous measurement of position and momentum

The simultaneous *perfect* measurement of $x$ and $p$ would violate complementarity.




Joint position and  measurement gives partial information on both the position and the momentum.

Position-momentum minimum uncertainty state.

# POM description of joint measurements

Probability density:

$$\wp(x_m, p_m) = Tr\left[\hat{\rho}\hat{\pi}(x_m, p_m)\right]$$

Minimum uncertainty states:

$$\left|x_m, p_m\right\rangle = \left(2\pi\sigma^2\right)^{-1/4}\int dx \exp\left[-\frac{(x-x_m)^2}{4\sigma^2} + ip_m x\right]\left|x\right\rangle$$

$$\frac{1}{2\pi\hbar}\iint dx_m dp_m \left|x_m, p_m\right\rangle\left\langle x_m, p_m\right| = \hat{\mathrm{I}}$$

This leads us to the POM elements:

$$\hat{\pi}(x_m, p_m) = \frac{1}{2\pi\hbar} |x_m, p_m\rangle\langle x_m, p_m|$$

The associated position probability distribution is

$$\wp(x_m) = \int dx \langle x|\hat{\rho}|x\rangle \exp\left[-\frac{(x-x_m)^2}{2\sigma^2}\right]$$

$$\Rightarrow \mathrm{Var}(x_m) = \Delta x^2 + \sigma^2$$

$$\& \;\; \mathrm{Var}(p_m) = \Delta p^2 + \frac{\hbar^2}{4\sigma^2}$$

Increased uncertainty is the price we pay for measuring *x* and *p*.

1. A bit about photons

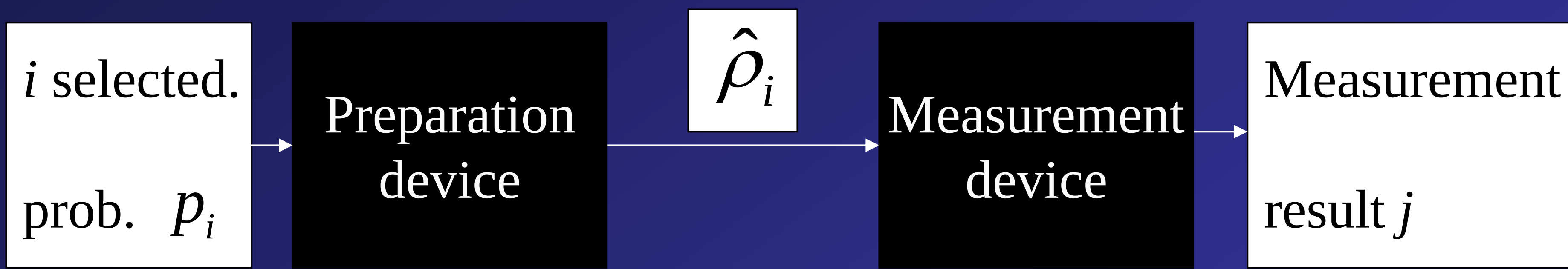
2. Optical polarisation

3. Generalised measurements

4. State discrimination
    - Minimum error
    - Unambiguous
    - Maximum confidence

## The communications problem

'Alice' prepares a quantum system in one of a set of $N$ possible signal states and sends it to 'Bob'




$$P(j|i) = \mathrm{Tr}\left(\hat{\pi}_j \hat{\rho}_i\right)$$

Bob is more interested in $$P(i|j) = \frac{\mathrm{Tr}\left(\hat{\pi}_j \hat{\rho}_i\right) p_i}{\mathrm{Tr}(\hat{\pi}_j \hat{\rho})}$$

In general, signal states will be non-orthogonal. No measurement can distinguish perfectly between such states.

Were it possible then there would exist a POM with

$$\langle\psi_1|\hat{\pi}_1|\psi_1\rangle = 1 = \langle\psi_2|\hat{\pi}_2|\psi_2\rangle$$
$$\langle\psi_2|\hat{\pi}_1|\psi_2\rangle = 0 = \langle\psi_1|\hat{\pi}_2|\psi_1\rangle$$

Completeness, positivity and $\langle\psi_1|\hat{\pi}_1|\psi_1\rangle = 1$

$$\Rightarrow \hat{\pi}_1 = |\psi_1\rangle\langle\psi_1| + \hat{A} \qquad \hat{A} \text{ positive and } \hat{A}|\psi_1\rangle = 0$$

$$\Rightarrow \langle\psi_2|\hat{\pi}_1|\psi_2\rangle = |\langle\psi_1|\psi_2\rangle|^2 + \langle\psi_2|\hat{A}|\psi_2\rangle \geq |\langle\psi_1|\psi_2\rangle|^2 \neq 0$$

What is the best we can do? Depends on what we mean by ‘best’.

## Minimum-error discrimination

We can associate each measurement operator $\hat{\pi}_i$ with a signal state $\hat{\rho}_i$. This leads to an error probability

$$P_e = 1 - \sum_{j=1}^{N} p_j Tr\left(\hat{\pi}_j \hat{\rho}_j\right)$$

Any POM that satisfies the conditions

$$\hat{\pi}_j (p_j \hat{\rho}_j - p_k \hat{\rho}_k) \hat{\pi}_k = 0 \qquad \forall j, k$$

$$\sum_{k=1}^{N} p_k \hat{\rho}_k \hat{\pi}_k - p_j \hat{\rho}_j \geq 0 \qquad \forall j$$

will minimise the probability of error.

For just two states, we require a von Neumann measurement with projectors onto the eigenstates of $p_1\hat{\rho}_1 - p_2\hat{\rho}_2$ with positive (1) and negative (2) eigenvalues:

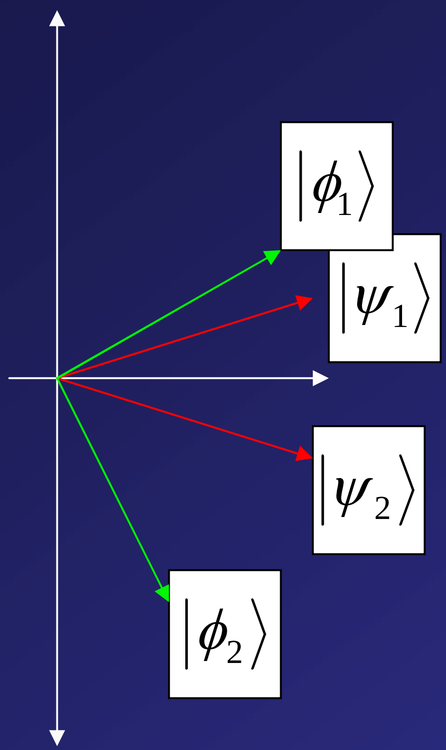
$$P_e^{\min} = \tfrac{1}{2}\left(1 - \mathrm{Tr}\left|p_1\hat{\rho}_1 - p_2\hat{\rho}_2\right|\right)$$

Consider for example the two pure qubit-states

$$|\psi_1\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$$

$$\langle\psi_1|\psi_2\rangle = \cos(2\theta)$$

$$|\psi_2\rangle = \cos\theta|0\rangle - \sin\theta|1\rangle$$




The minimum error is achieved by measuring in the orthonormal basis spanned by the states $|\phi_1\rangle$ and $|\phi_2\rangle$.

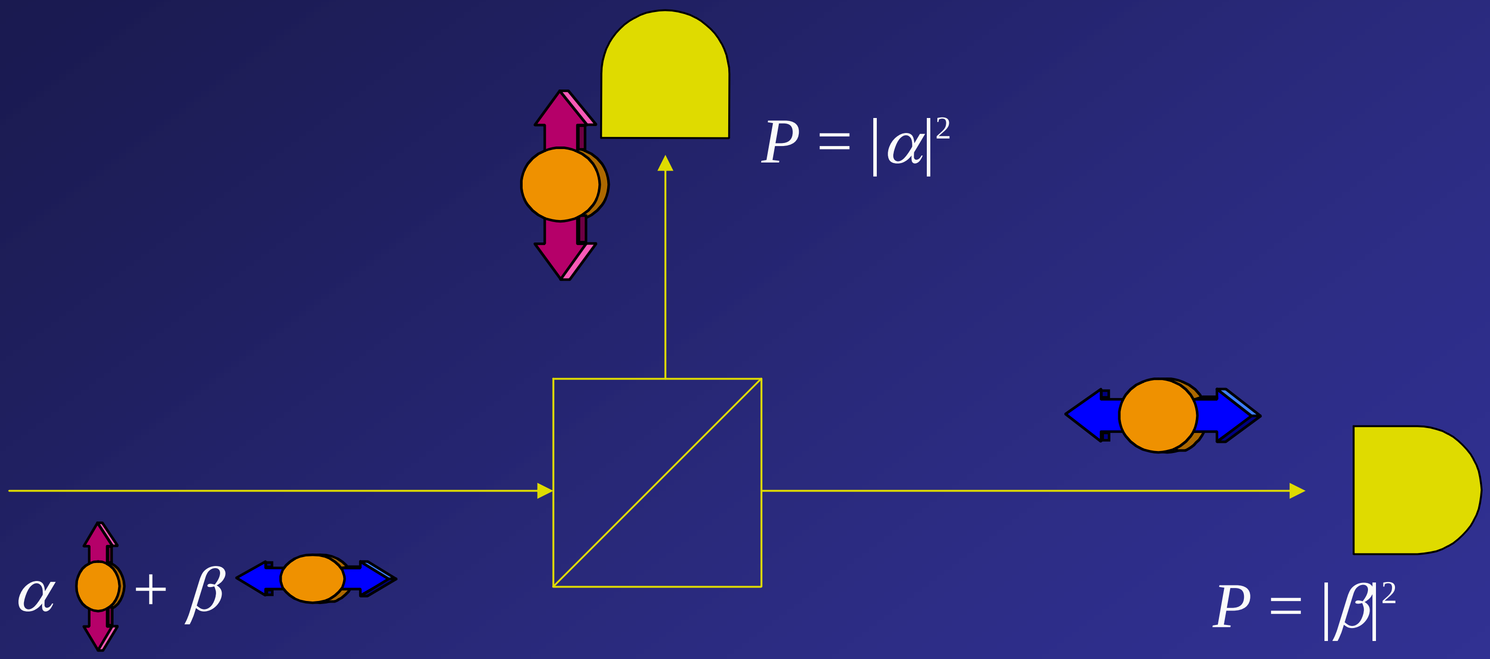
We associate $|\phi_1\rangle$ with $|\psi_1\rangle$ and $|\phi_2\rangle$ with $|\psi_2\rangle$:

$$P_e = p_1 |\langle \psi_1 | \phi_2 \rangle|^2 + p_2 |\langle \psi_2 | \phi_1 \rangle|^2$$

The minimum error is the Helstrom bound

$$P_e^{\min} = \tfrac{1}{2}\left[1 - \left(1 - 4 p_1 p_2 |\langle \psi_1 | \psi_2 \rangle|^2\right)^{1/2}\right]$$

# A single photon only gives one "click"




But this is all we need to discriminate between our two states with minimum error.

A more challenging example is the ‘trine ensemble’ of three equiprobable states:

$$|\psi_1\rangle = -\tfrac{1}{2}\left(|0\rangle + \sqrt{3}|1\rangle\right) \qquad p_1 = \tfrac{1}{3}$$

$$|\psi_2\rangle = -\tfrac{1}{2}\left(|0\rangle - \sqrt{3}|1\rangle\right) \qquad p_2 = \tfrac{1}{3}$$

$$|\psi_3\rangle = |0\rangle \qquad p_3 = \tfrac{1}{3}$$

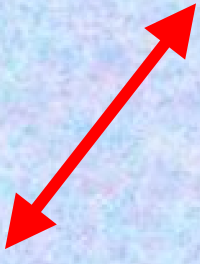
It is straightforward to confirm that the minimum-error conditions are satisfied by the three probability operators

$$\hat{\pi}_i = \tfrac{2}{3}|\psi_i\rangle\langle\psi_i|$$

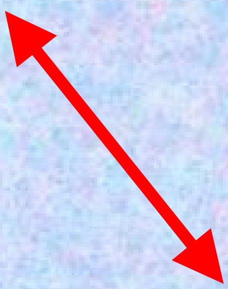
# Simple example - the trine states

## Three symmetric states of photon polarisation

$$|\psi_3\rangle = |\leftrightarrow\rangle$$



$$|\psi_2\rangle = \frac{-|\leftrightarrow\rangle + \sqrt{3}|\updownarrow\rangle}{2}$$



$$|\psi_1\rangle = \frac{|\leftrightarrow\rangle + \sqrt{3}|\updownarrow\rangle}{2}$$

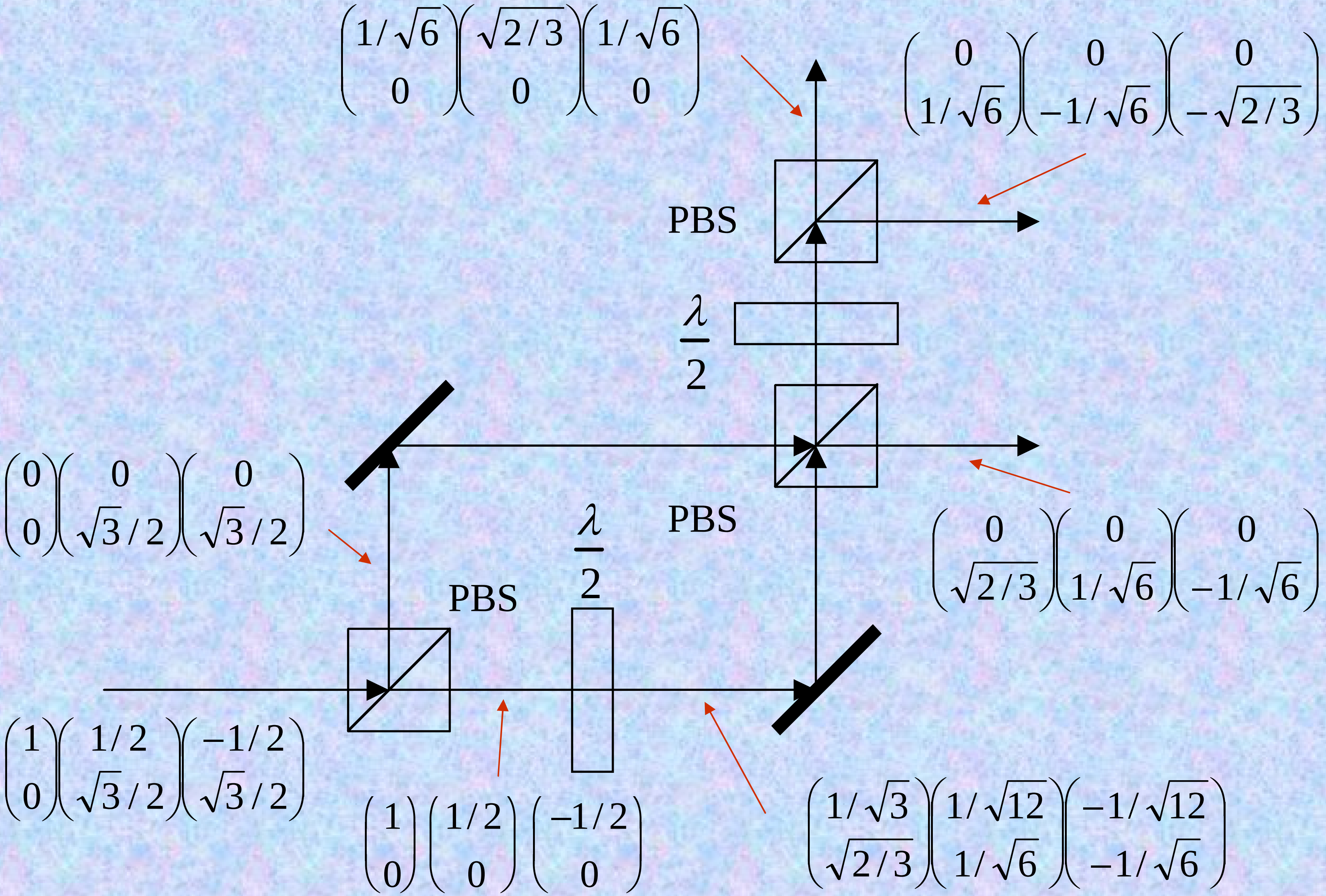
Minimum error probability is 1/3.

This corresponds to a POM with elements

$$\hat{\pi}_j = \frac{2}{3}|\psi_j\rangle\langle\psi_j|$$

How can we do a polarisation measurement with these three possible results?

# Polarisation interferometer - Sasaki *et al*, Clarke *et al*.



# Unambiguous discrimination

The existence of a minimum error does not mean that error-free or unambiguous state discrimination is impossible. A von Neumann measurement with

$$\hat{P}_1 = |\psi_1\rangle\langle\psi_1| \qquad \hat{P}_{\bar{1}} = |\psi_1^{\perp}\rangle\langle\psi_1^{\perp}|$$

will give unambiguous identification of $|\psi_2\rangle$ :

result $\quad \bar{1} \quad \Rightarrow \quad 2$ error-free

result $\quad 1 \quad \Rightarrow \quad ?$ inconclusive

There is a more symmetrical approach with

$$\hat{\pi}_1 = \frac{1}{1+|\langle\psi_1|\psi_2\rangle|}|\psi_2^{\perp}\rangle\langle\psi_2^{\perp}|$$

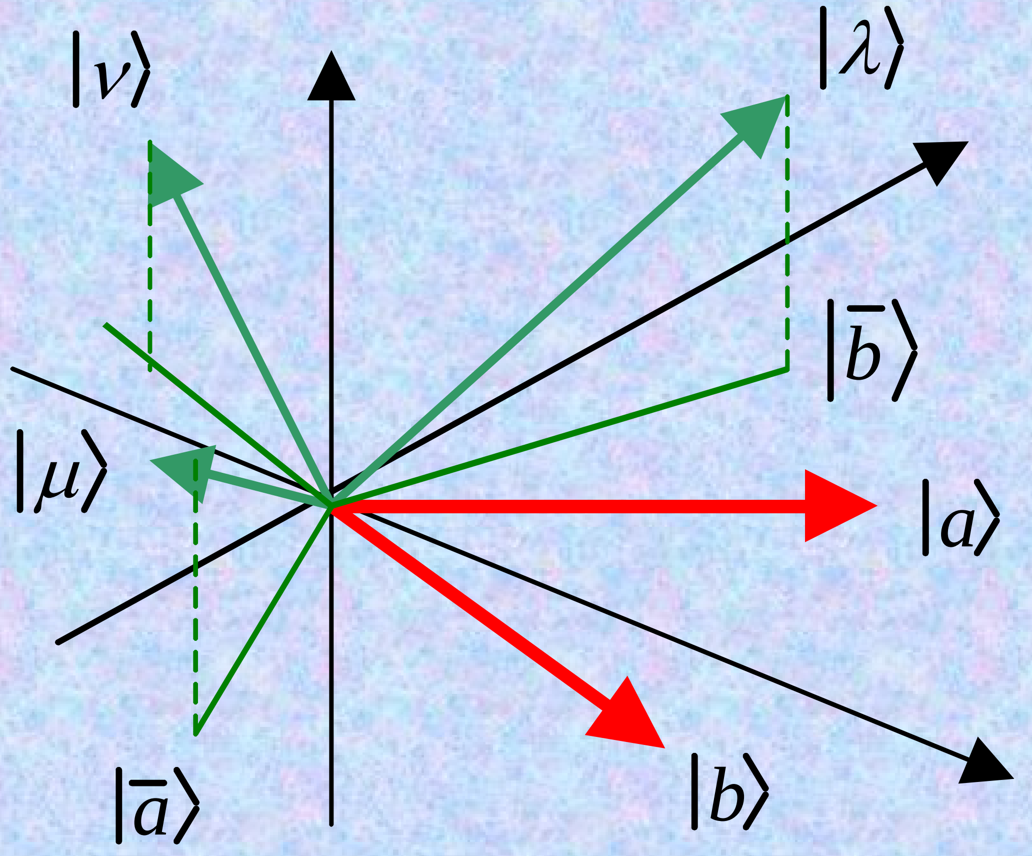
$$\hat{\pi}_2 = \frac{1}{1+|\langle\psi_1|\psi_2\rangle|}|\psi_1^{\perp}\rangle\langle\psi_1^{\perp}|$$

$$\hat{\pi}_? = \hat{\mathrm{I}} - \hat{\pi}_1 - \hat{\pi}_2$$

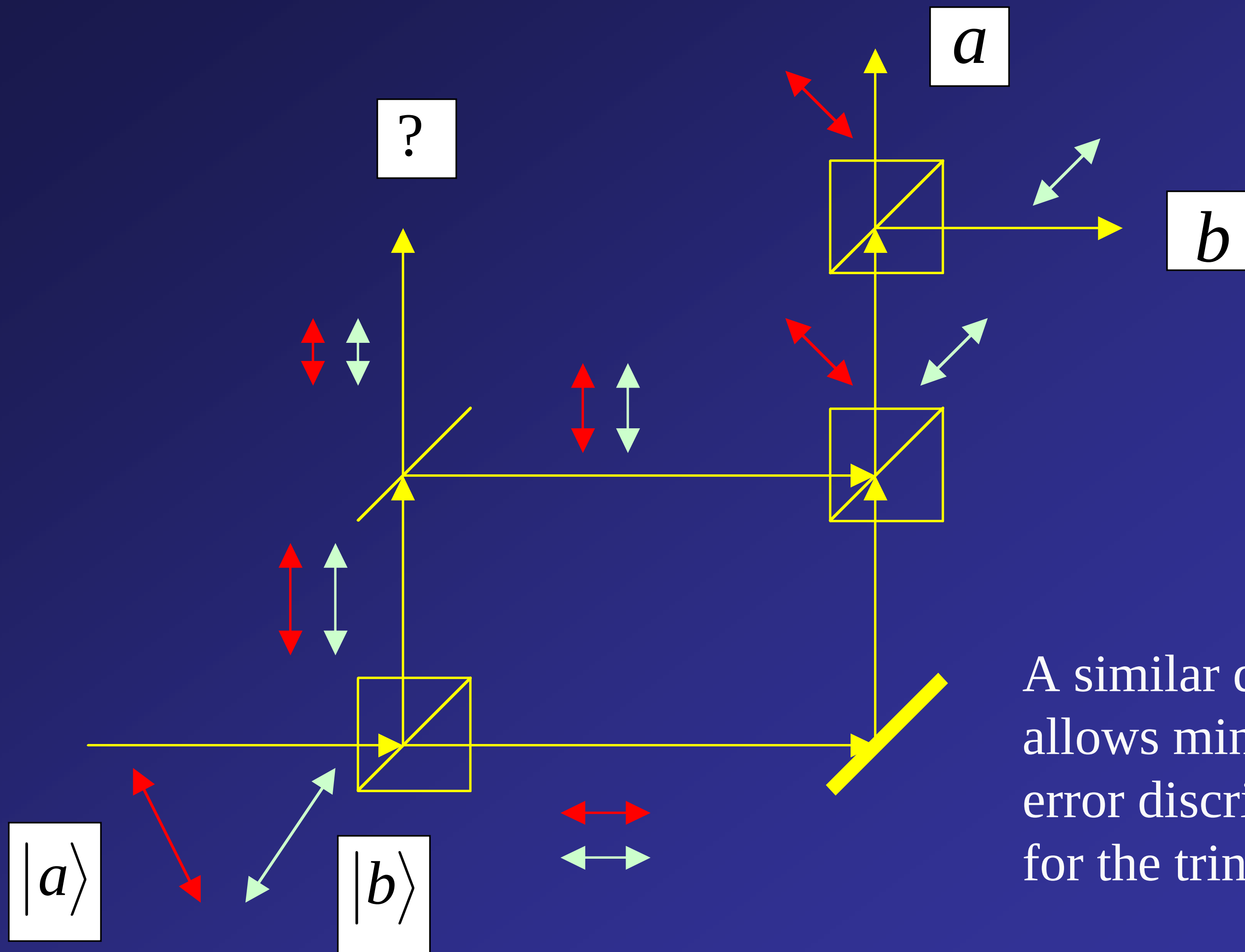
| | Result 1 | Result 2 | Result ? |
|---|---|---|---|
| State $|\psi_1\rangle$ | $1-|\langle\psi_1|\psi_2\rangle|$ | 0 | $|\langle\psi_1|\psi_2\rangle|$ |
| State $|\psi_2\rangle$ | 0 | $1-|\langle\psi_1|\psi_2\rangle|$ | $|\langle\psi_1|\psi_2\rangle|$ |

How can we understand the IDP measurement?

Consider an extension into a 3D state-space



Unambiguous state discrimination - Huttner *et al*, Clarke *et al*.




A similar device allows minimum error discrimination for the trine states.

Maximum confidence measurements seek to maximise the conditional probabilities

$$P(\psi_i \mid \omega_i)$$

for each state.

*For unambiguous discrimination these are all 1.*

Bayes' theorem tells us that

$$P(\psi_i \mid \omega_i) = \frac{p_i P(\omega_i \mid \psi_i)}{P(\omega_i)} = \frac{p_i \langle \psi_i | \hat{\pi}_i | \psi_i \rangle}{\sum_k p_k \langle \psi_k | \hat{\pi}_i | \psi_k \rangle}$$

so the largest values of those give us maximum confidence.

The solution we find is

$$\hat{\pi}_i \propto \hat{\rho}^{-1}\hat{\rho}_j\hat{\rho}^{-1}$$

where

$$\hat{\rho}_j = |\psi_j\rangle\langle\psi_j|$$

$$\hat{\rho} = \sum_i p_i\hat{\rho}_i$$

Croke *et al* Phys. Rev. Lett. **96**, 070401 (2006)

# Example

- 3 states in a 2-dimensional space

$$
\begin{aligned}
|\Psi_0\rangle &= \cos\theta|0\rangle + \sin\theta|1\rangle \\
|\Psi_1\rangle &= \cos\theta|0\rangle + e^{2\pi i/3}\sin\theta|1\rangle \\
|\Psi_2\rangle &= \cos\theta|0\rangle + e^{-2\pi i/3}\sin\theta|1\rangle
\end{aligned}
$$

- Maximum Confidence Measurement:

$$\hat{\Pi}_j = a_j|\phi_j\rangle\langle\phi_j|$$

$$
\begin{aligned}
|\phi_0\rangle &= \sin\theta|0\rangle + \cos\theta|1\rangle \\
|\phi_1\rangle &= \sin\theta|0\rangle + e^{2\pi i/3}\cos\theta|1\rangle \\
|\phi_2\rangle &= \sin\theta|0\rangle + e^{-2\pi i/3}\cos\theta|1\rangle
\end{aligned}
$$

- Inconclusive outcome needed

$$\hat{\Pi}_? = (1 - \tan^2\theta)|0\rangle\langle 0|$$

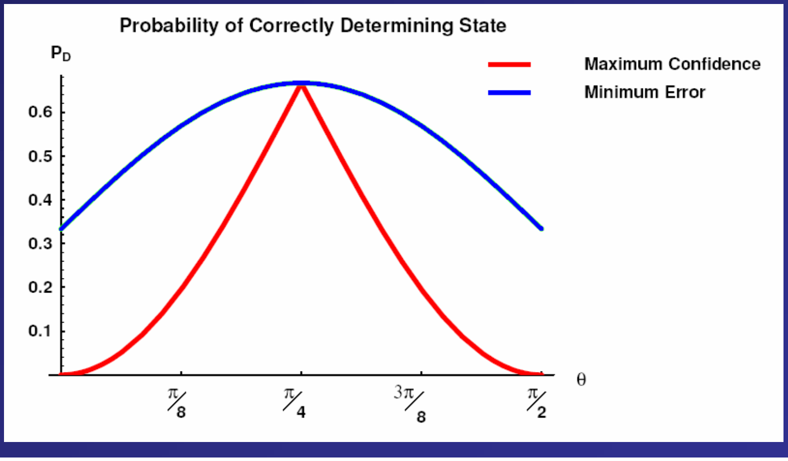
# Optimum probabilities

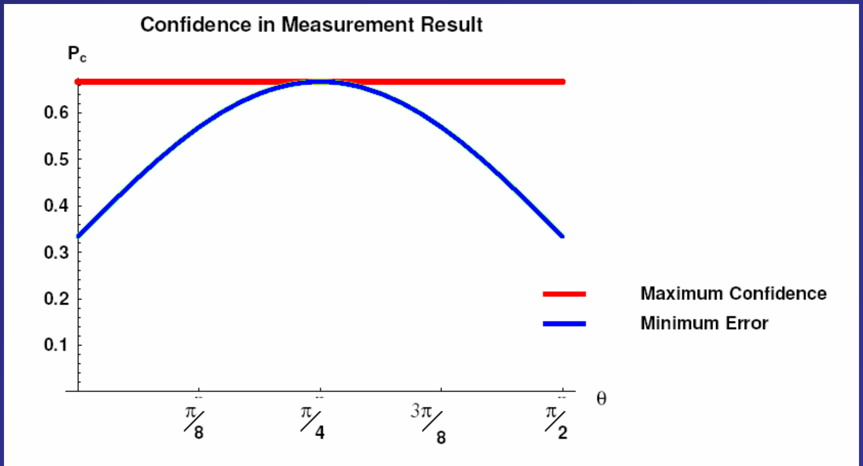
- Probability of correctly determining state maximised for minimum

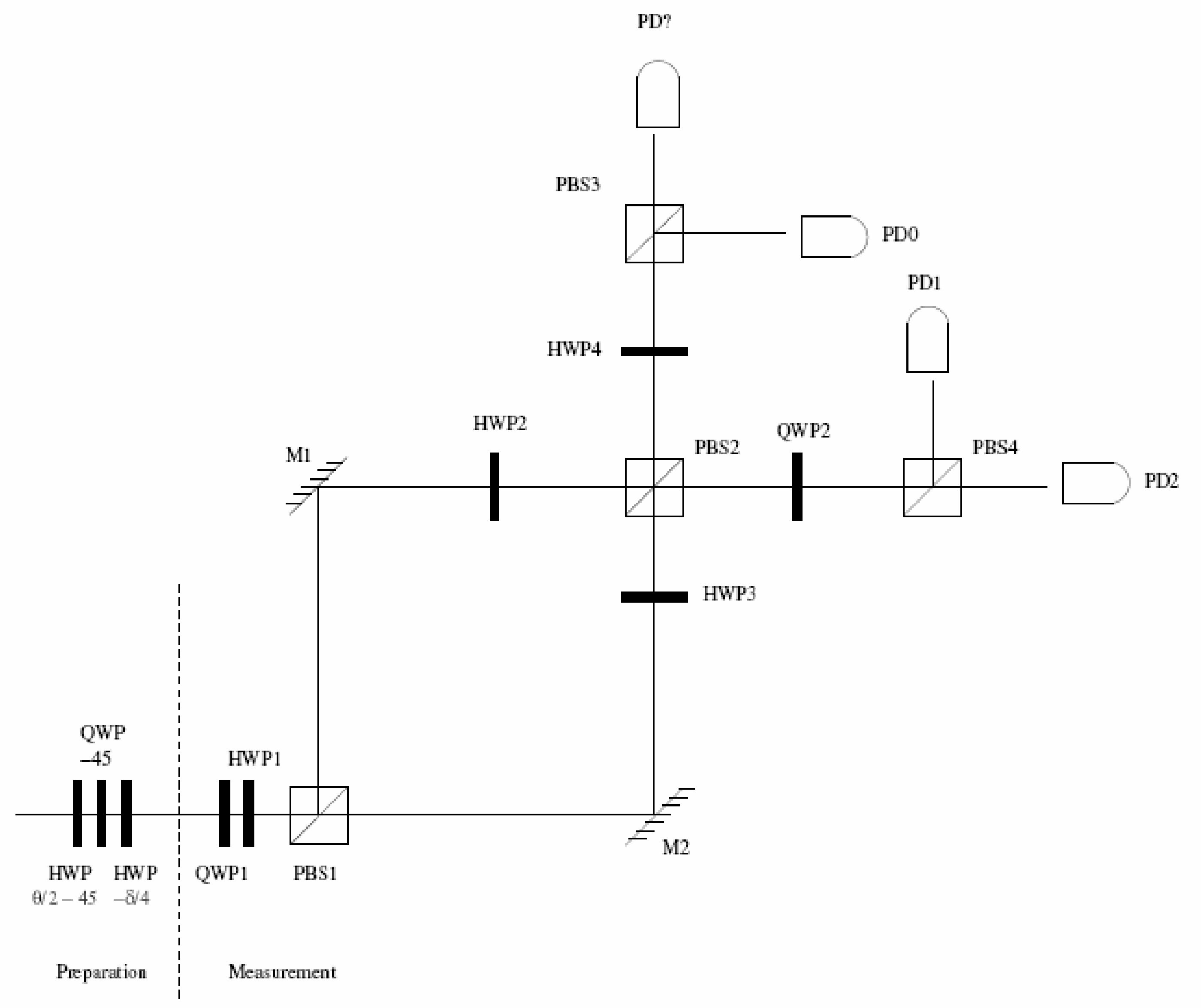
$$P_D = \sum_j P(\psi_j)P(\omega_j|\psi_j)$$

- Probability that result obtained is correct maximised by maximum confidence measurement:

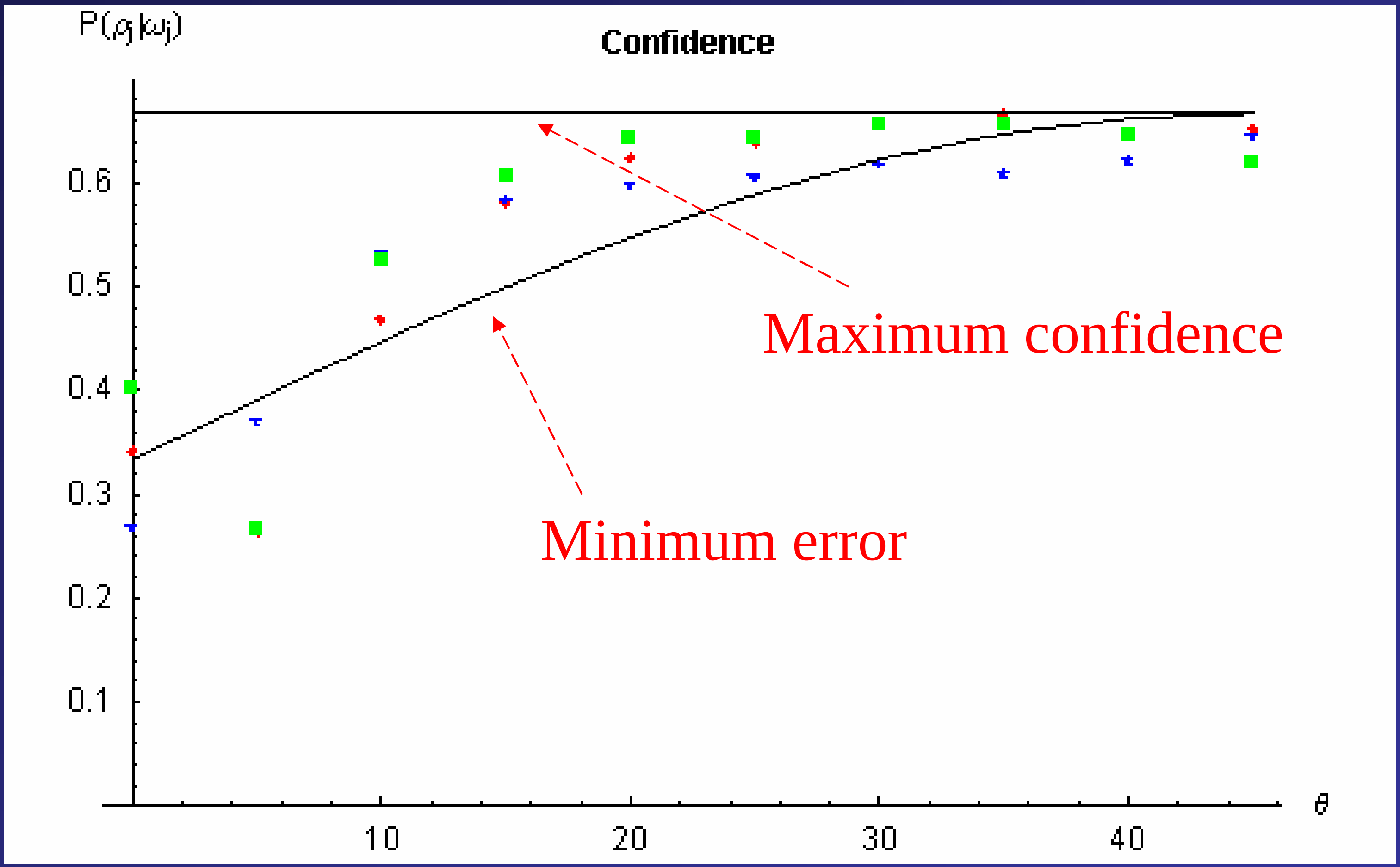
$$P(\psi_j|\omega_j) = \frac{P(\psi_j)P(\omega_j|\psi_j)}{P(\omega_j)}$$




PD?
PBS3
PD0
PD1
HWP4
HWP2
M1
PBS2
QWP2
PBS4
PD2
HWP3
QWP
-45
HWP1
M2
HWP
θ/2 – 45
HWP
–δ/4
QWP1
PBS1
Preparation
Measurement

# Results:



Confidence
0.6
0.5
0.4
0.3
0.2
0.1
10
20
30
40
θ
Maximum confidence
Minimum error

# Conclusions

- Photons have played a central role in the development of quantum theory and the quantum theory of light continues to provide surprises.

- True single photons are hard to make but are, perhaps, the ideal carriers of quantum information.

- It is now possible to demonstrate a variety of measurement strategies which realise optimised POMs

- The subject of quantum optics also embraces atoms, ions molecules and solids ...