Classical and quantum structures

QNet I & QDay III - Glasgow

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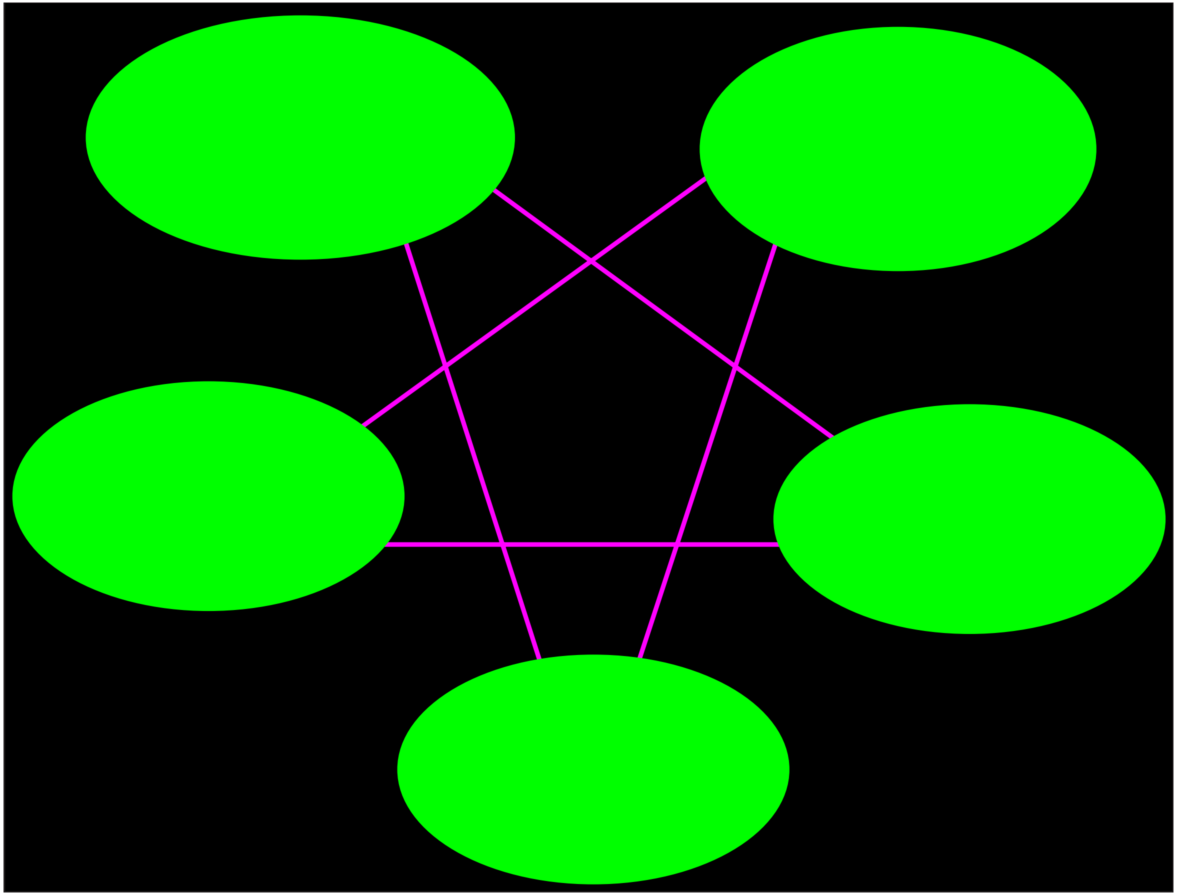
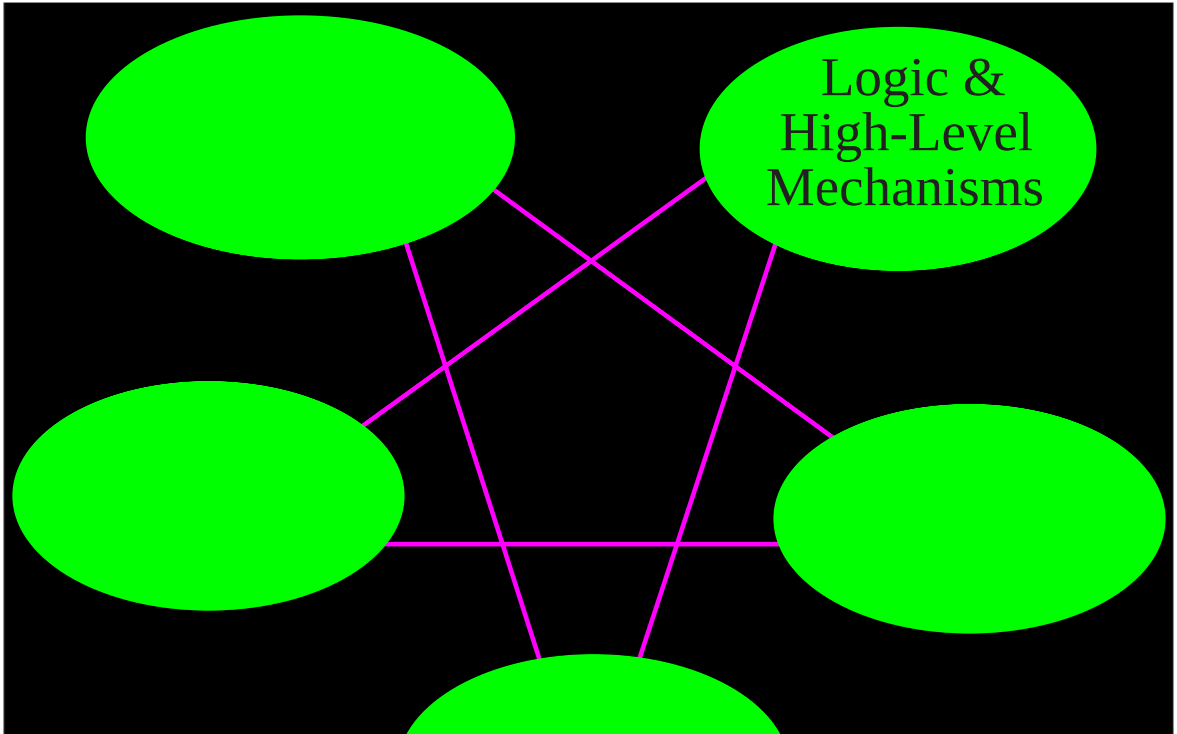
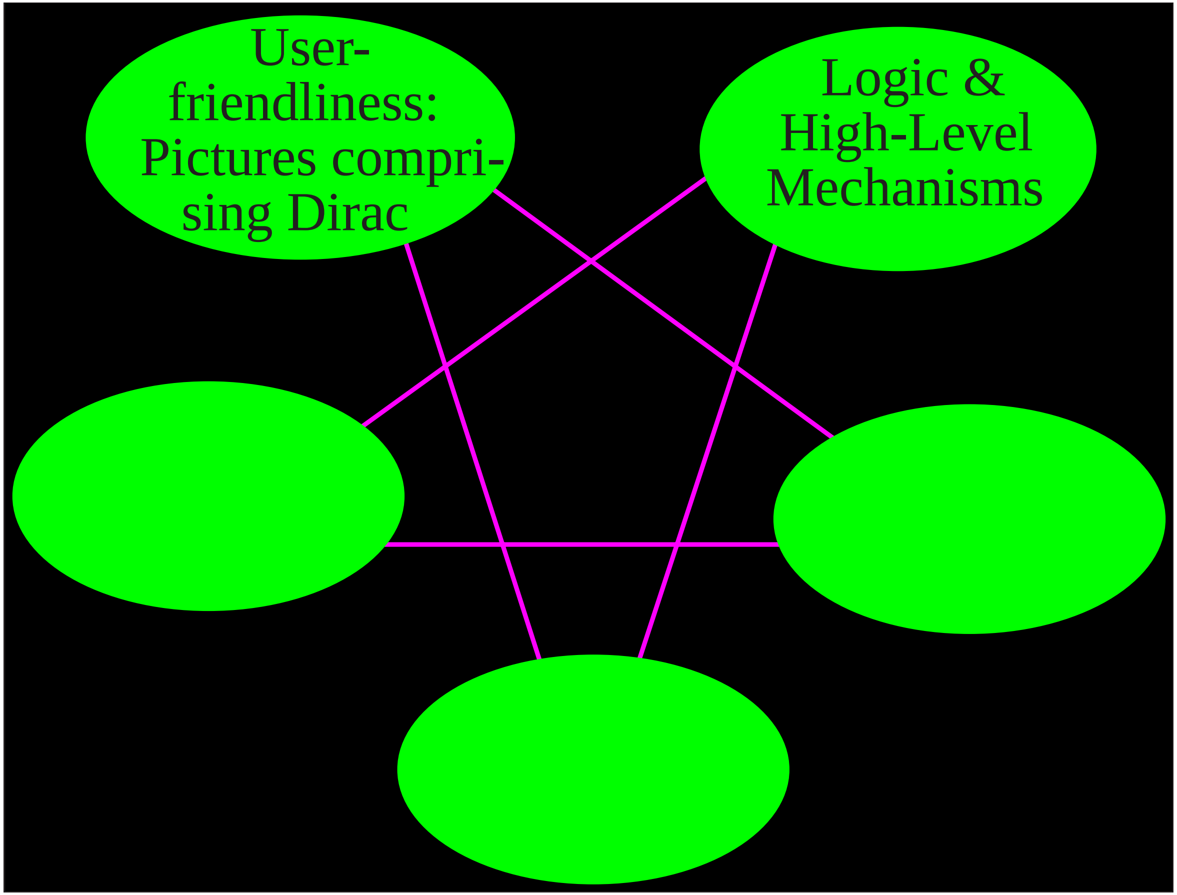
se10.comlab.ox.ac.uk:8080/BobCoecke/Home_en.html

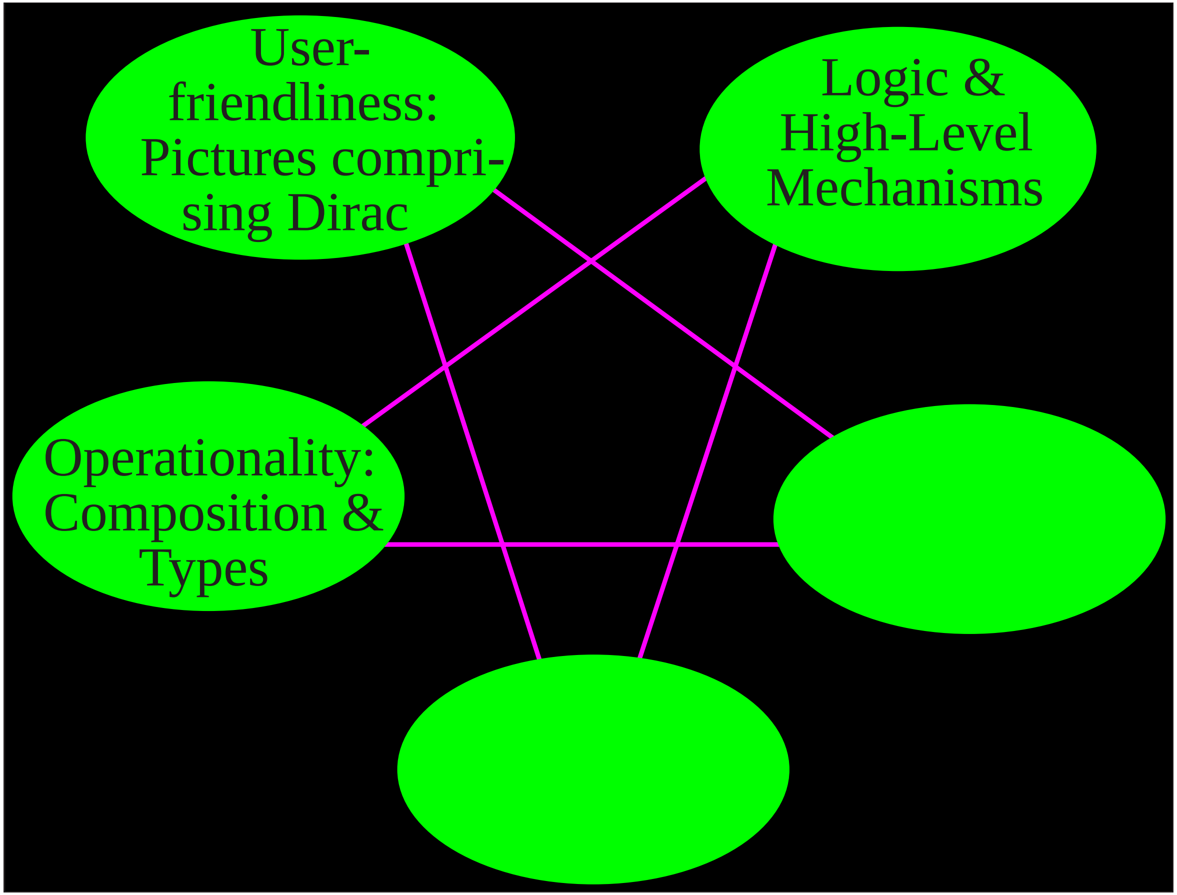
Work with Eric Paquette and Dusko Pavlovic, extending initial work with Samson Abramsky and Selinger's elaboration thereon.

ULTIMATE GOALS




Logic &
High-Level
Mechanisms


User-friendliness: Pictures comprising Dirac
Logic & High-Level Mechanisms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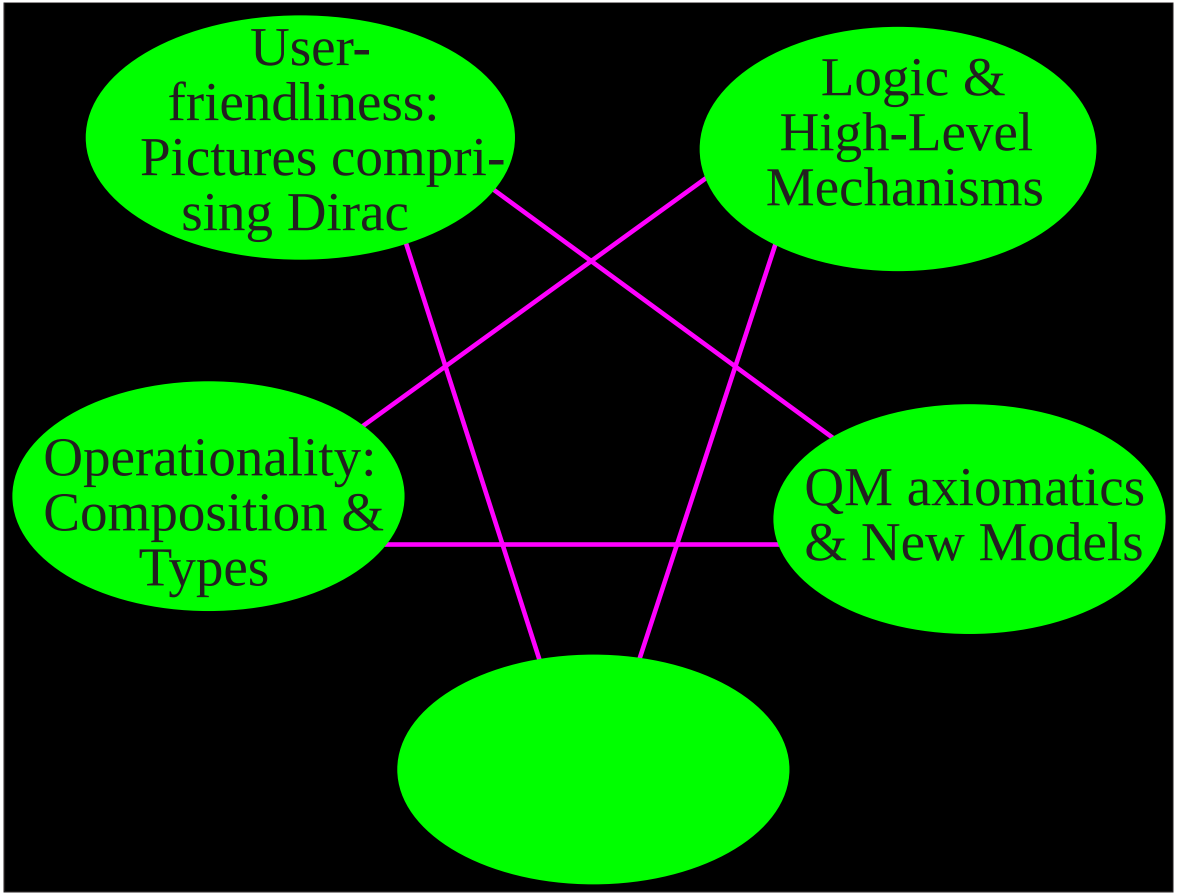
Operationality: Composition & Types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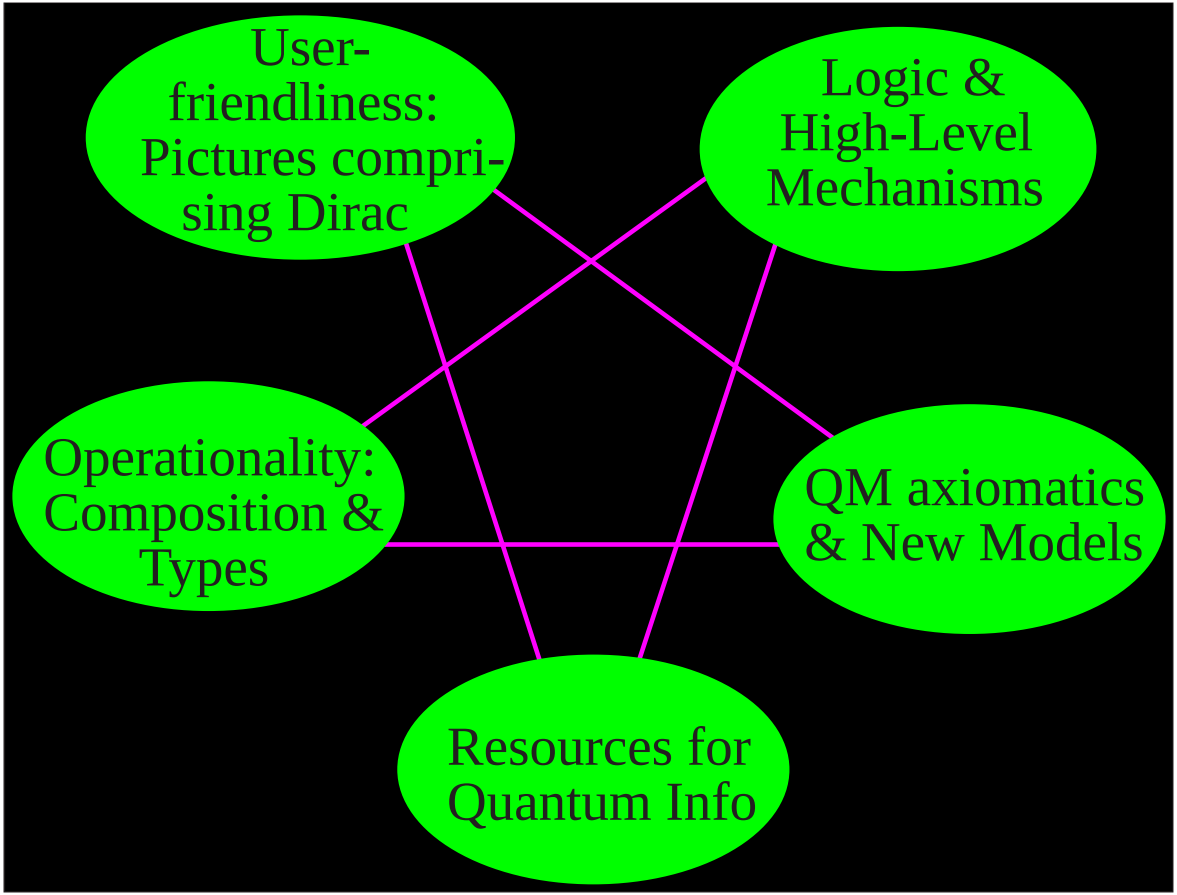
QM axiomatics & New Models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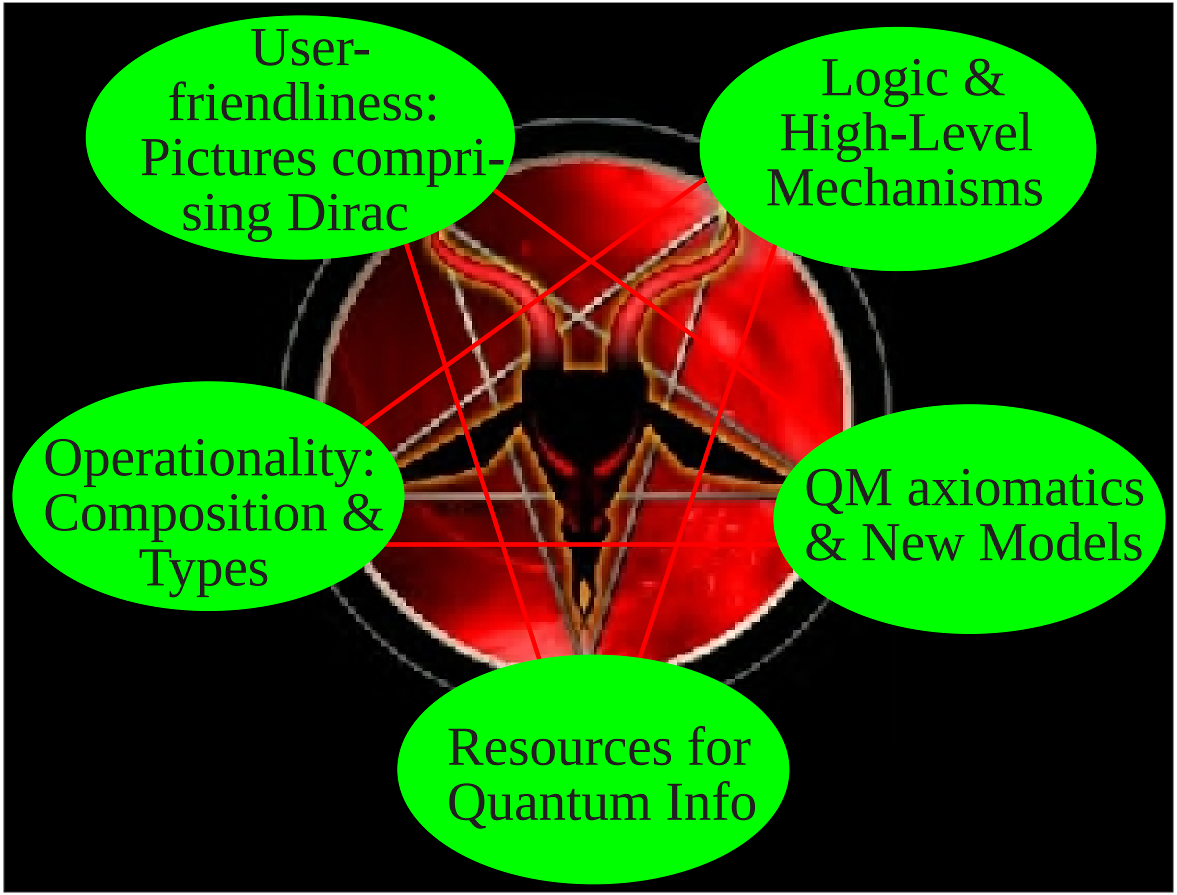
Resources for Quantum Info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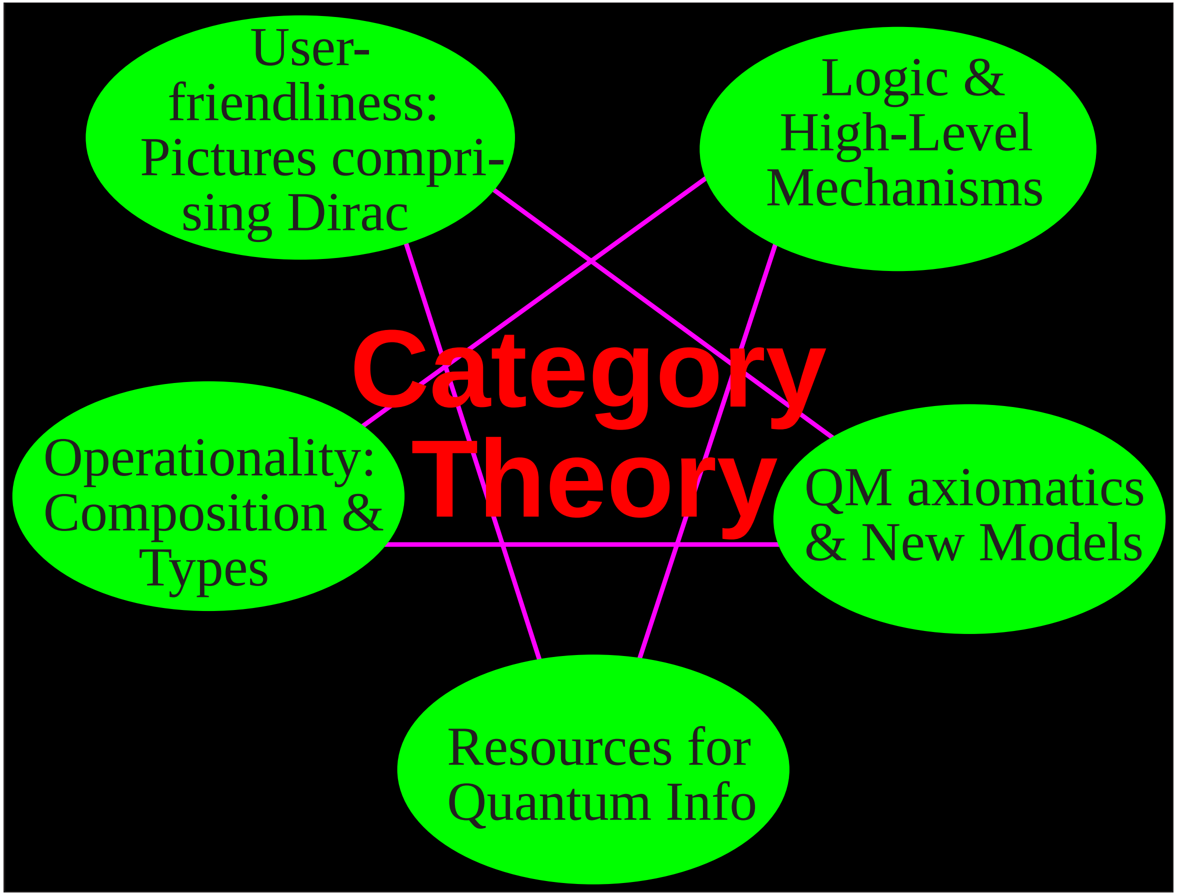
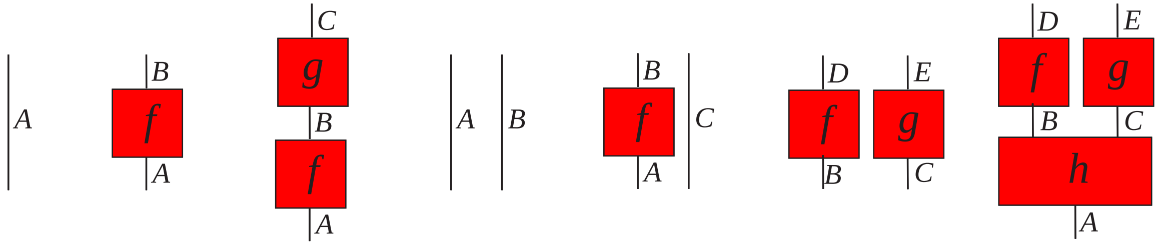

User-friendliness: Pictures comprising Dirac
Logic & High-Level Mechanisms
Category Theory
Operationality: Composition & Types
QM axiomatics & New Models
Resources for Quantum Info

BACKGROUND STRUCTURE

(Penrose, Joyal-Street, Freyd-Yetter, Turaev, ...)

Symmetric Monoidal Category

$$1_A \quad f \quad g \circ f \quad 1_A \otimes 1_B \quad f \otimes 1_C \quad f \otimes g \quad (f \otimes g) \circ h$$



Symmetric Monoidal Category

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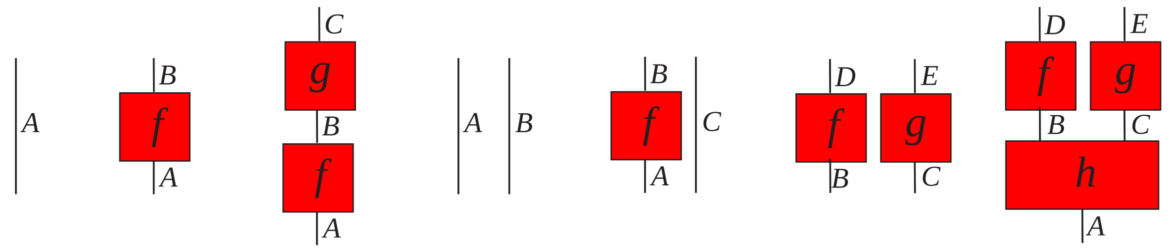
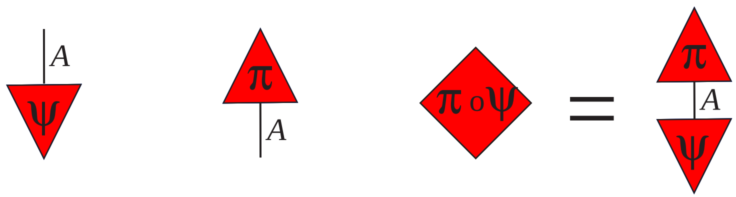
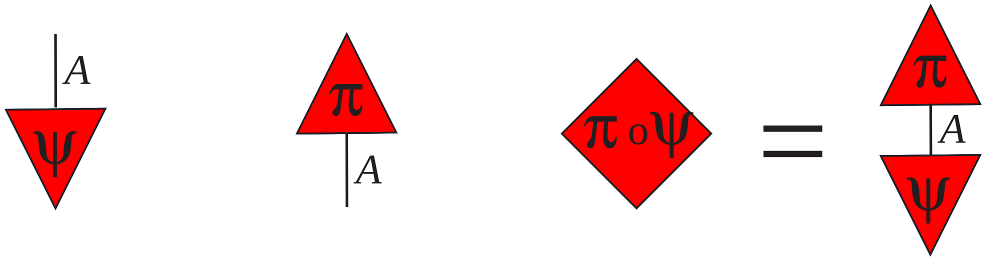
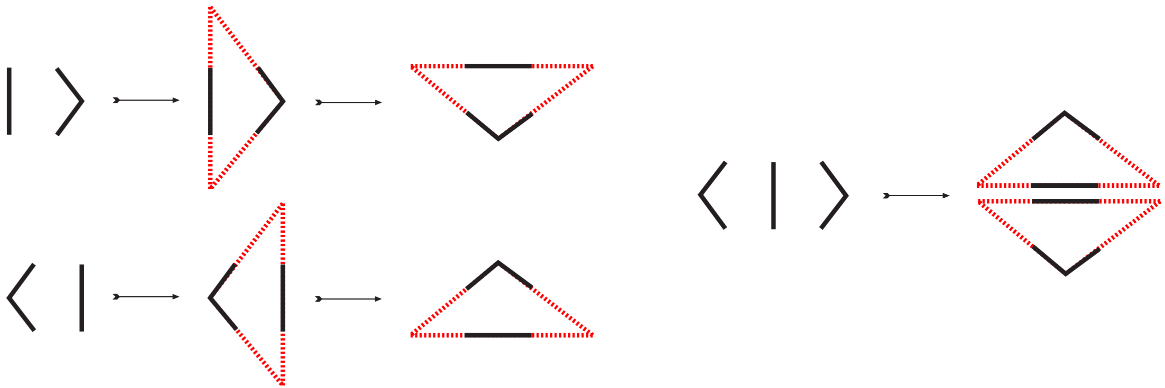


$\psi : \mathrm{I} \longrightarrow A \qquad \pi : A \longrightarrow \mathrm{I} \qquad \pi \circ \psi : \mathrm{I} \longrightarrow \mathrm{I}$



Symmetric Monoidal Category





"ket": $|\psi\rangle$

"bra": $\langle\psi|$

"bra-ket": $\langle\psi|\phi\rangle \in \mathbb{C}$

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"bra-ket": $\langle\psi|\phi\rangle \in \mathbb{C}$

"ket-bra": $|\psi\rangle\langle\psi|$ $\quad \bar{c}c \cdot |\psi\rangle\langle\psi| = |c\cdot\psi\rangle\langle c\cdot\psi|$

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"probability": $\langle\phi|\psi\rangle\langle\psi|\phi\rangle = \langle\phi|\mathrm{P}_\psi(\phi)\rangle$

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"mixed states": $\sum_i w_i|\psi_i\rangle\langle\psi_i| \neq |\phi\rangle\langle\phi|$

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“probability”: $\langle\phi|\psi\rangle\langle\psi|\phi\rangle = \langle\phi|\mathrm{P}_\psi(\phi)\rangle$

“mixed states”: $\sum_i w_i|\psi_i\rangle\langle\psi_i| \neq |\phi\rangle\langle\phi|$

“basis”: $\{|0\rangle, |1\rangle, \ldots, |n{-}1\rangle\}$ & $|ij\rangle := |i\rangle\otimes|j\rangle$

PRACTICING PHYSICS
Physical System
Physical Operation

PROGRAMMING
Data Types
Programs

LOGIC & PROOF THEORY
Propositions
Proofs

COOKING
Vegetables, meet, fish, spices, mayonaise
Growing, breeding, catching, cutting, mixing, eating

Symmetric Monoidal †-Category

$$f : A \to B \qquad \longleftrightarrow \qquad f^\dagger : B \to A$$

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$$f : A \to B \qquad \longleftrightarrow \qquad f^\dagger : B \to A$$

Most important non-†-cats can be fitted within a †-cat,
e.g. **Set** into **Rel**, **FStoch** into $\mathbf{Mat}_{\mathbb{R}^+}$, ...

QUANTUM STRUCTURE

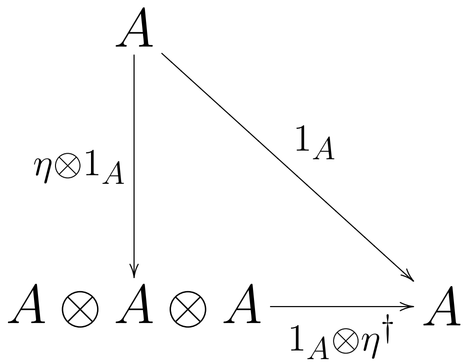
(Abramsky-Coecke 2004)

Object with (pure$_{\neg C}$) quantum structure

A pair

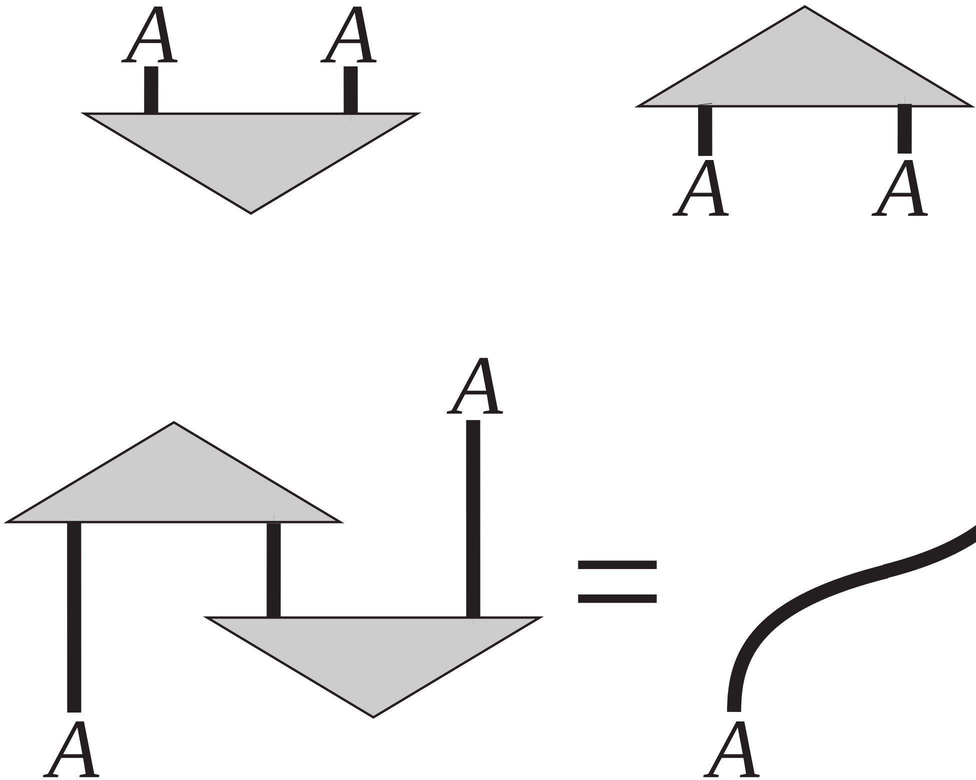
$$(A\,,\eta : \mathrm{I} \to A \otimes A)$$

such that



commutes. (†-compactness)

Object with (pure$_{\neg c}$) quantum structure



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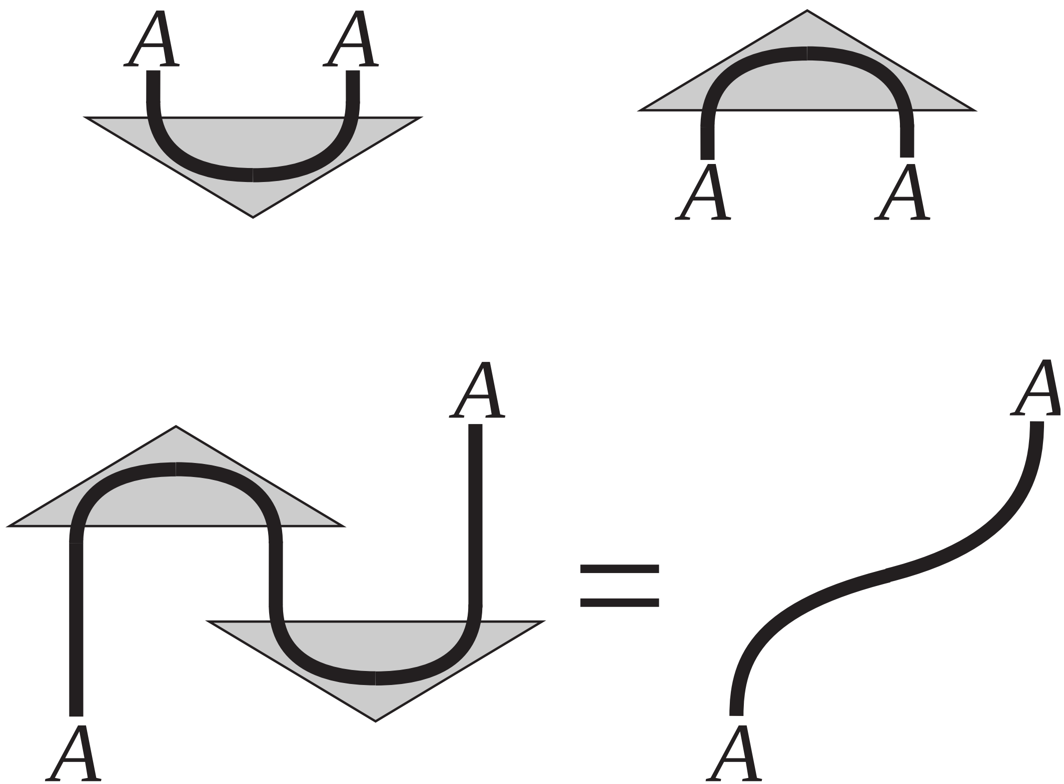
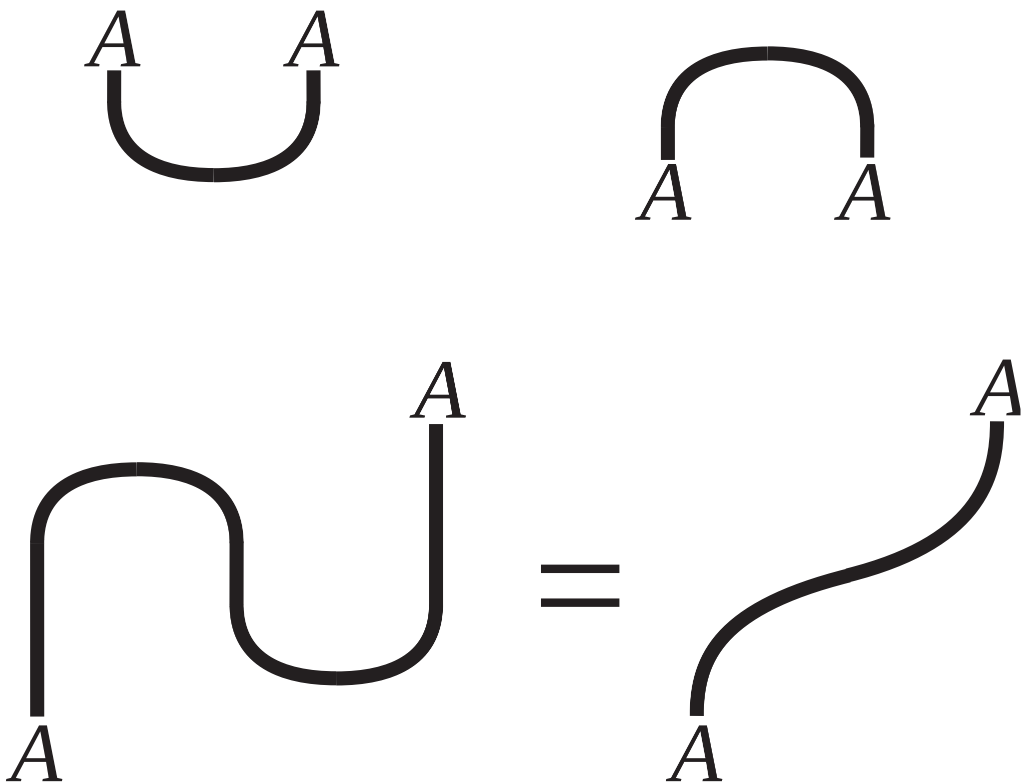
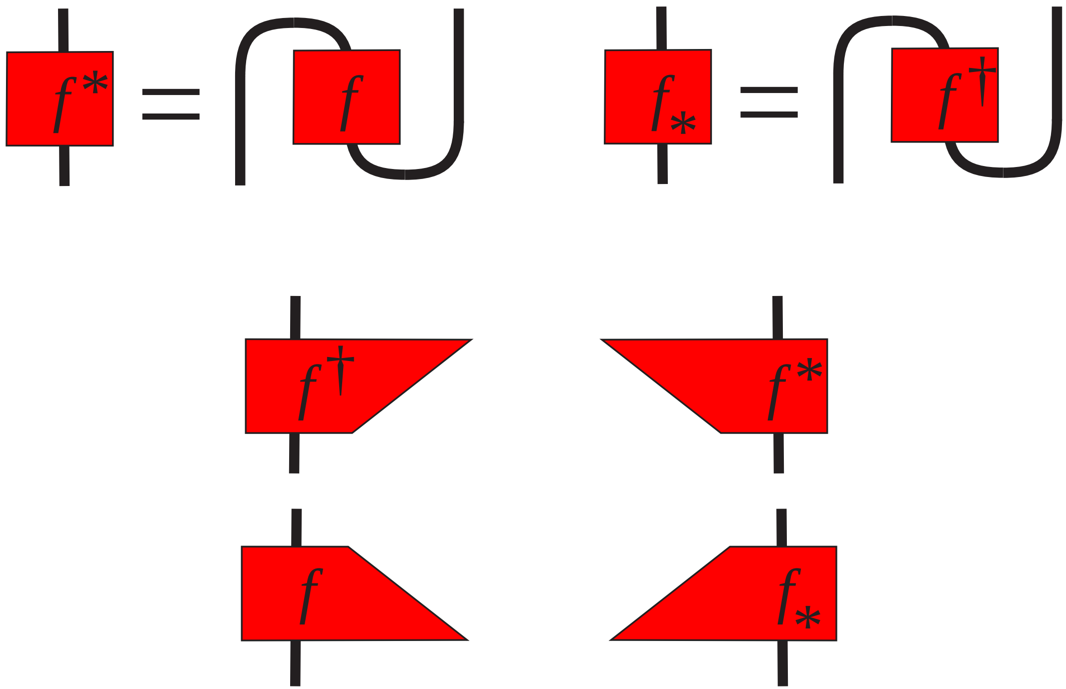


Object with (pure$_{\neg C}$) quantum structure



four-fold duality



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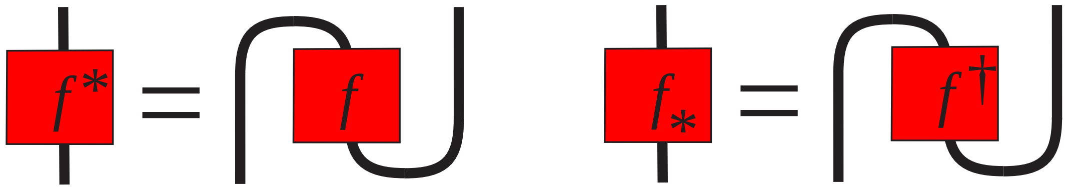
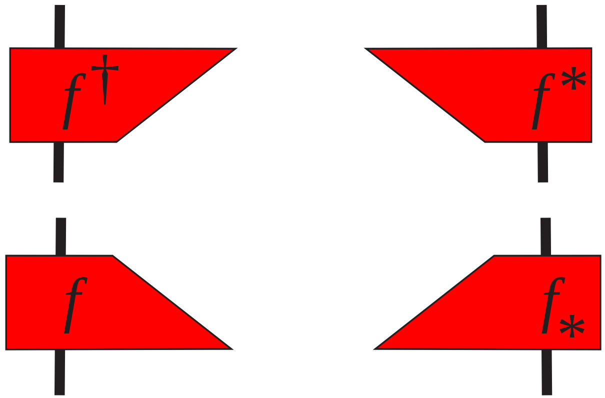
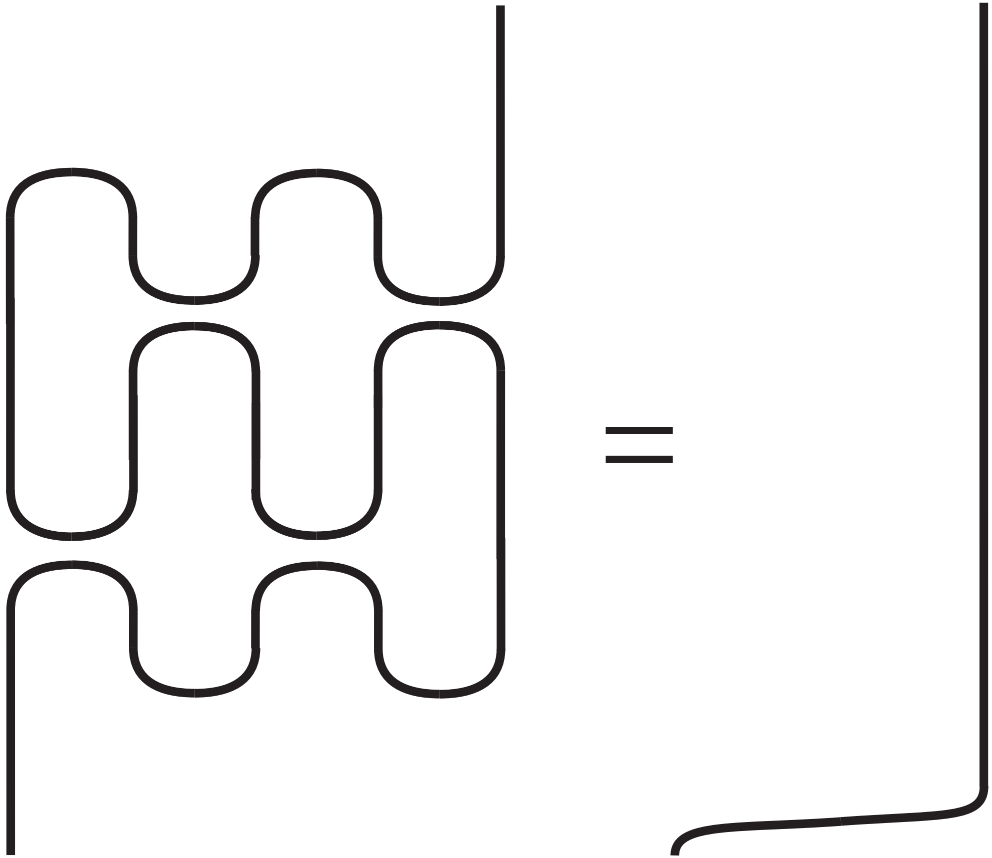


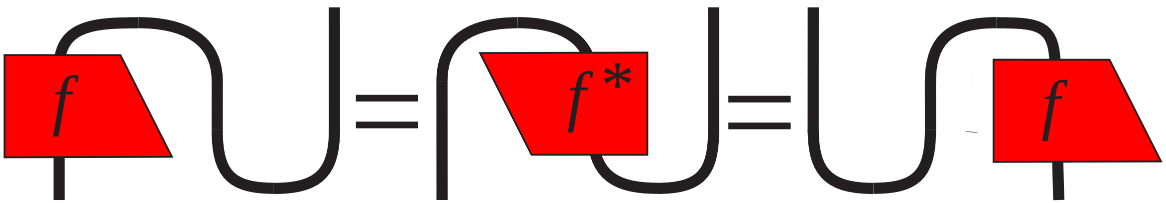


In **FdHilb**: $f^* \sim$ transposed & $f_* \sim$ conjugated

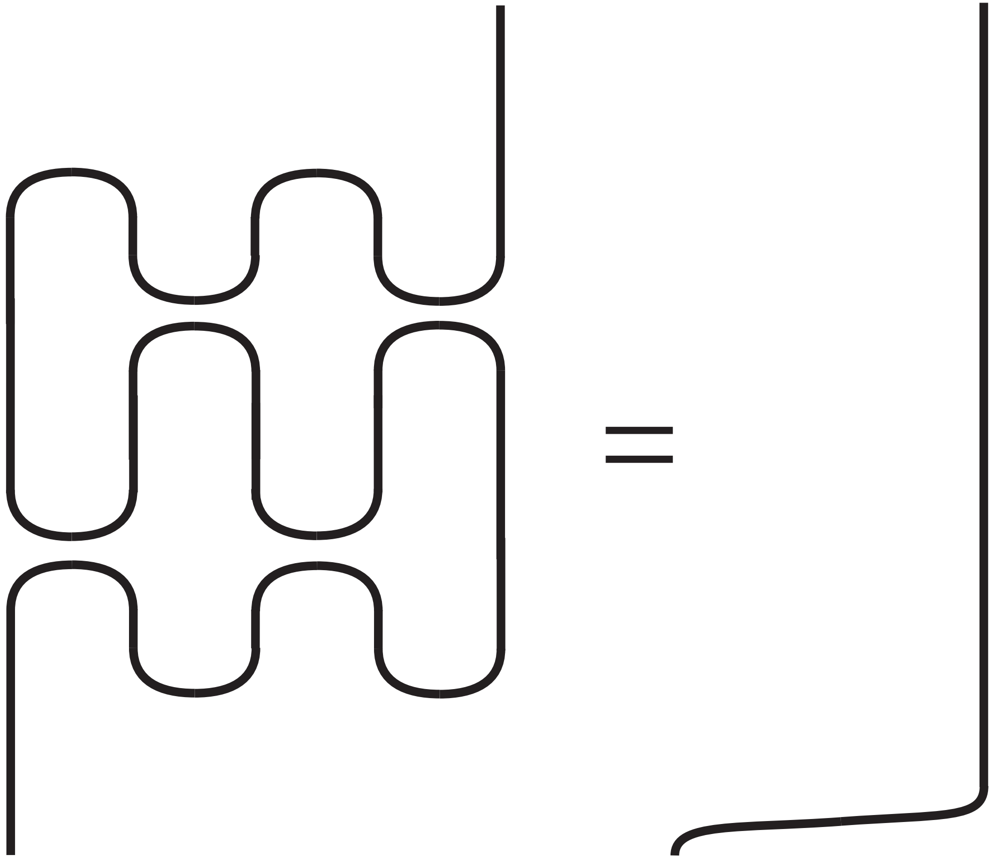
“Clean” normalization theorem



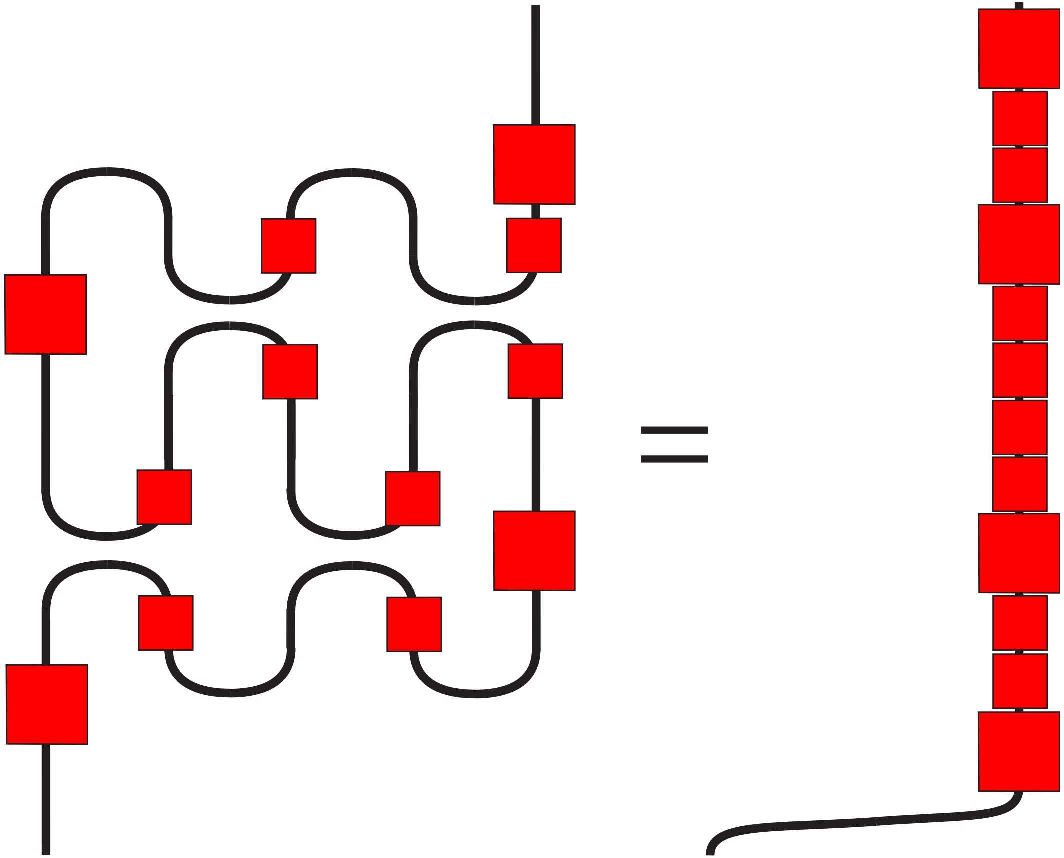
“Sliding” boxes



“Clean” normalization theorem



“Decorated” normalization theorem



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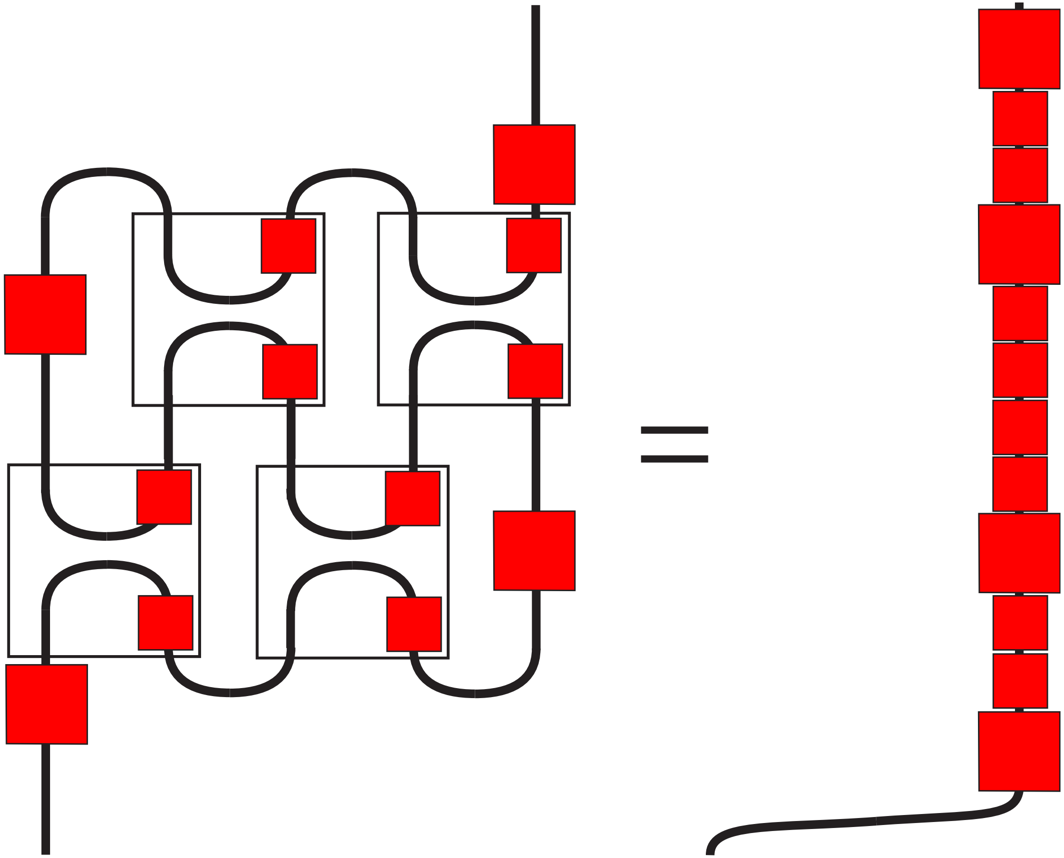

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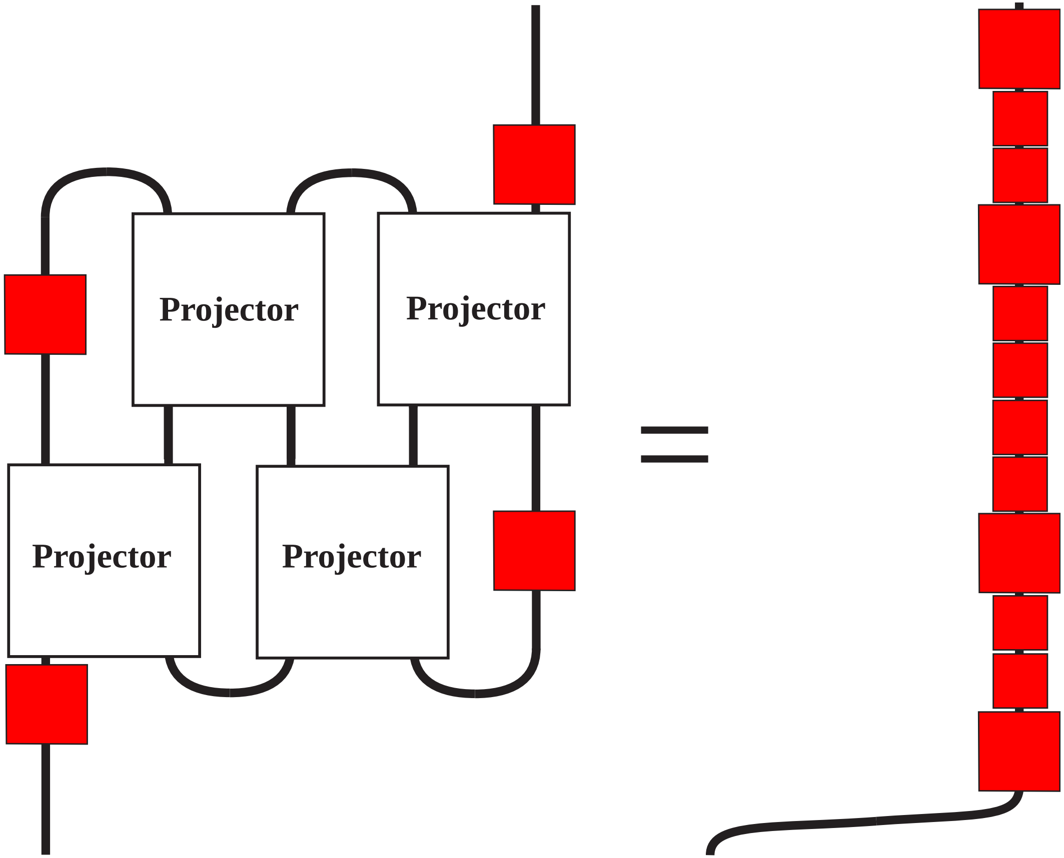
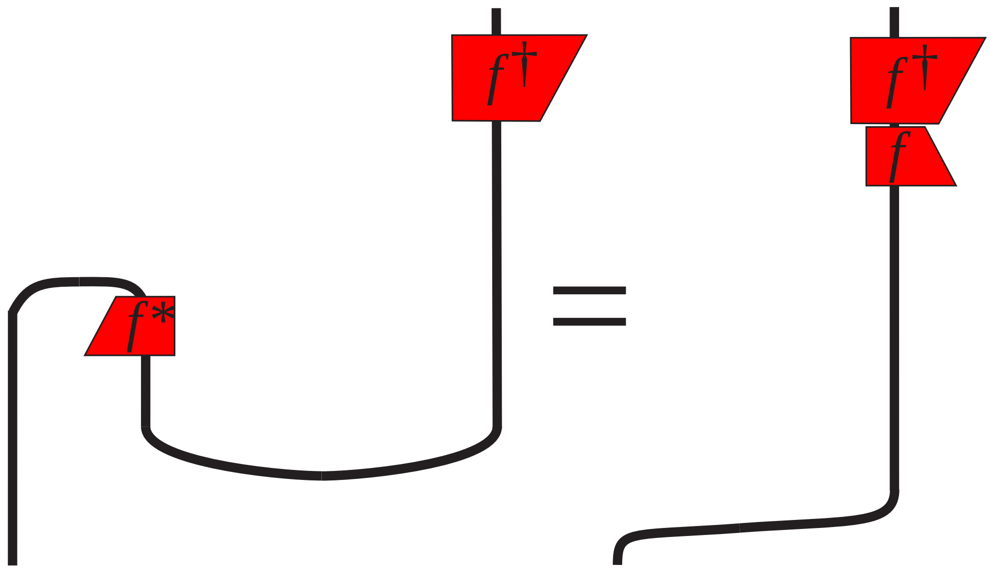


Applying "decorated" normalization 1



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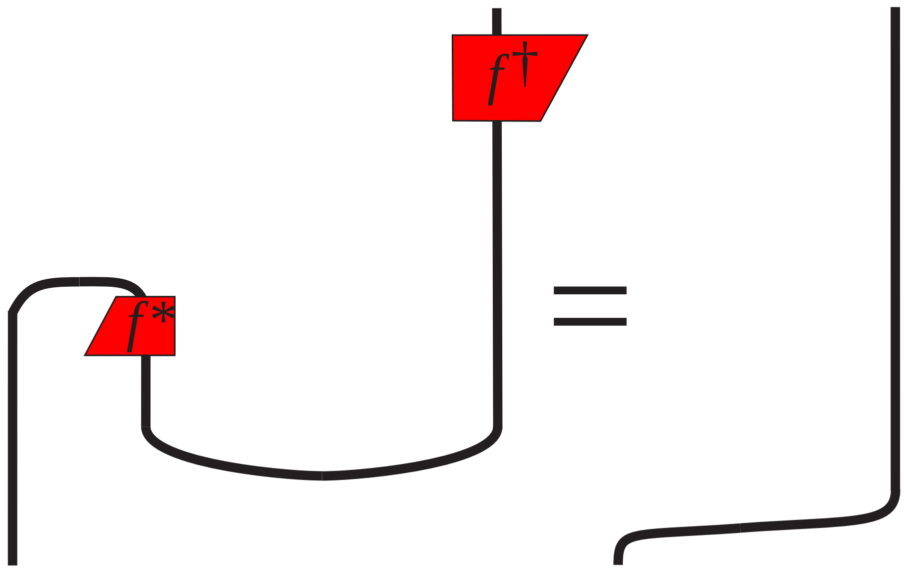

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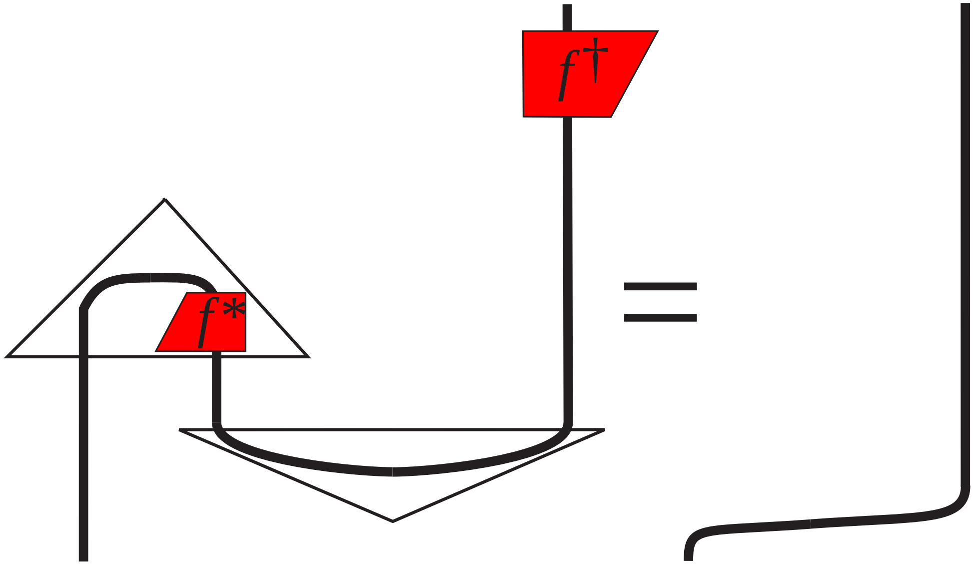

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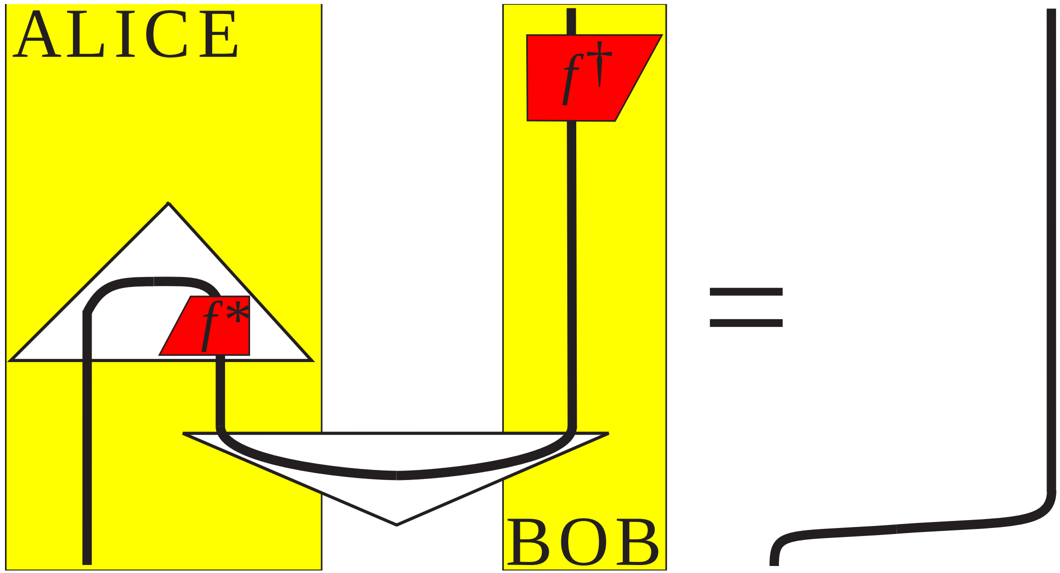
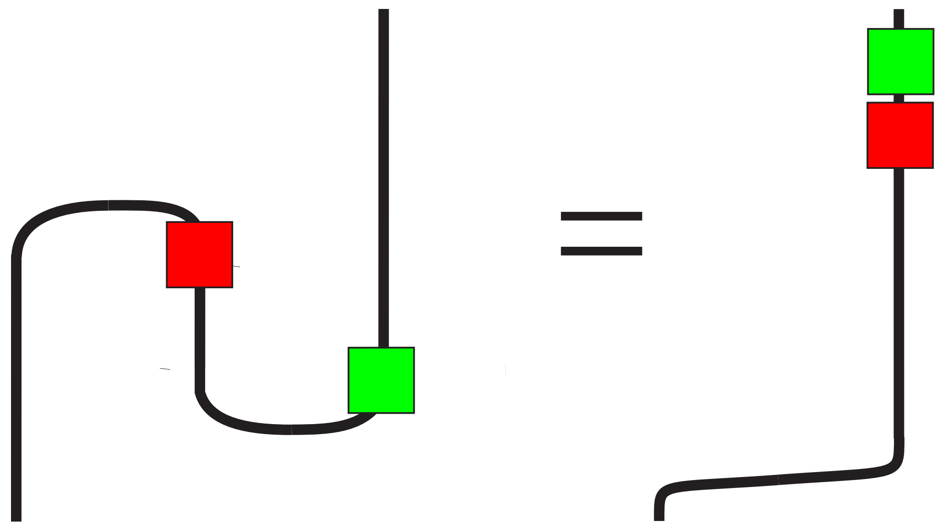
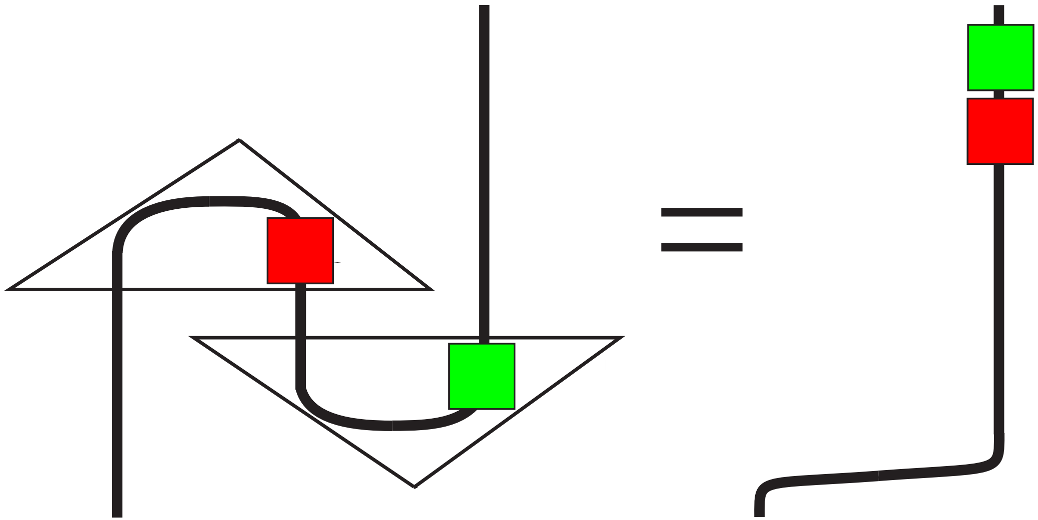


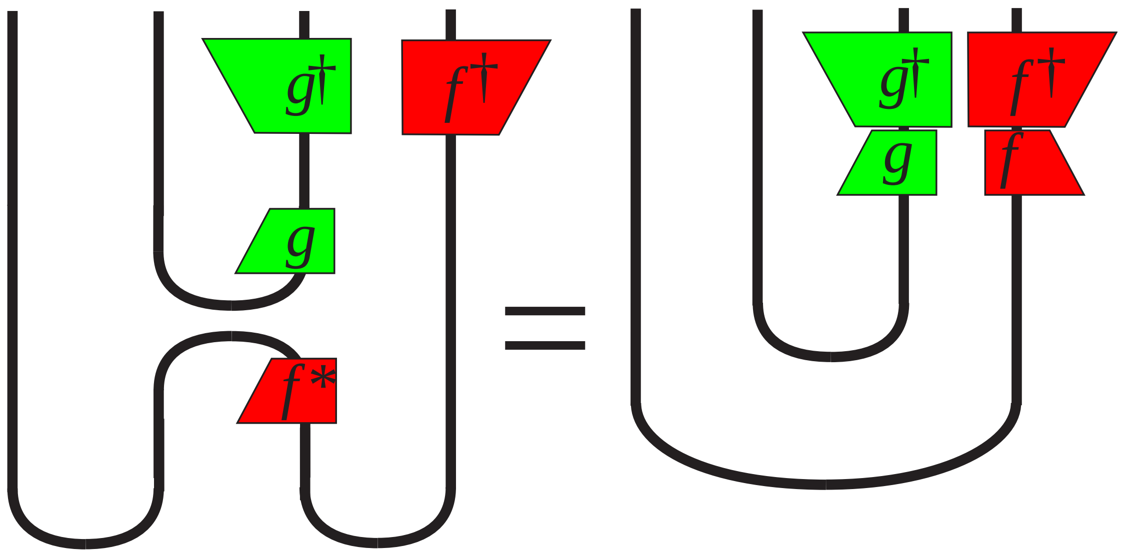
Applying "decorated" normalization 2



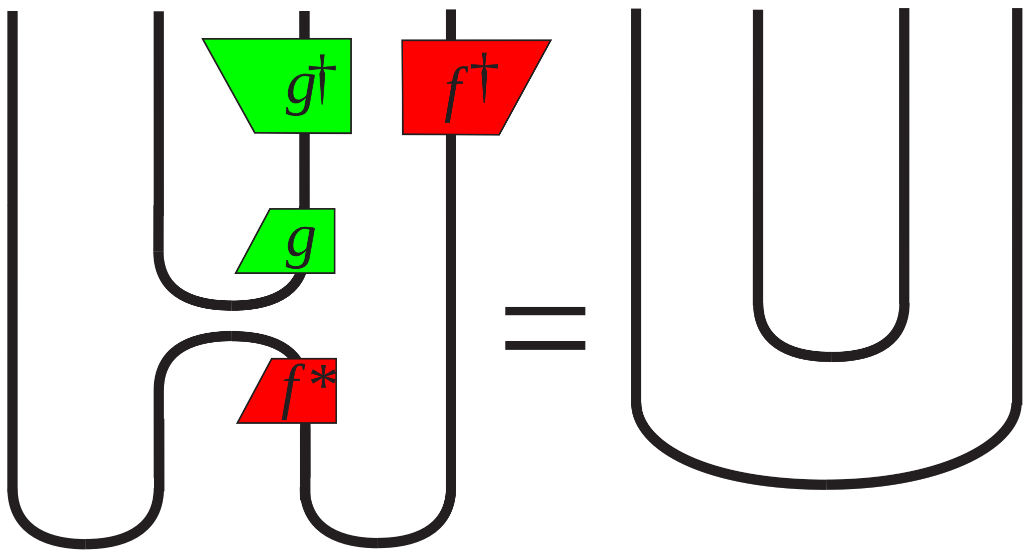
Applying "decorated" normalization 2



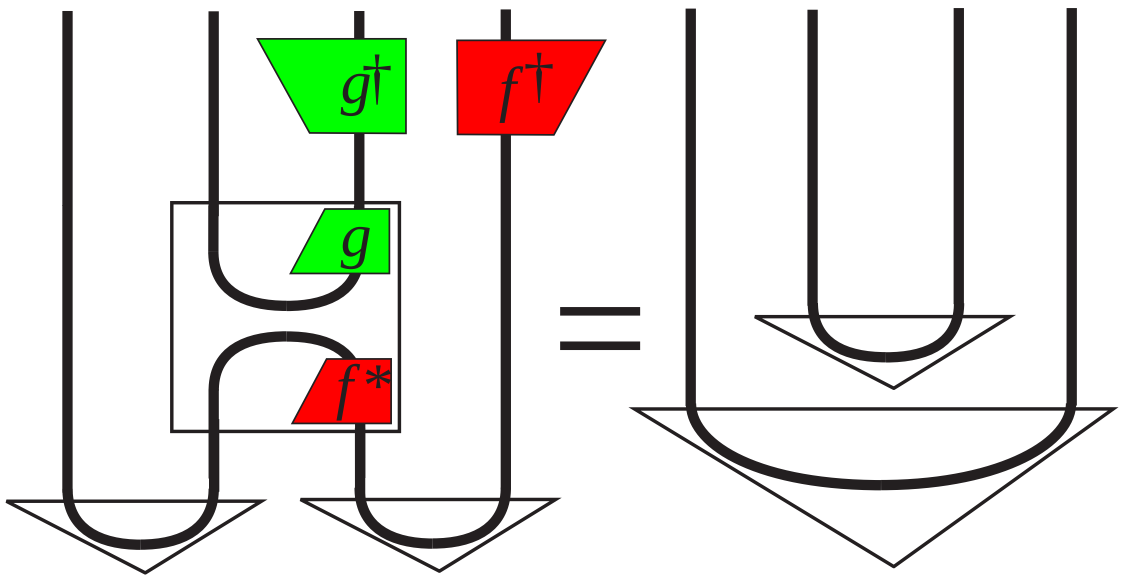
Applying “decorated” normalization 3



Applying "decorated" normalization 3



Applying "decorated" normalization 3



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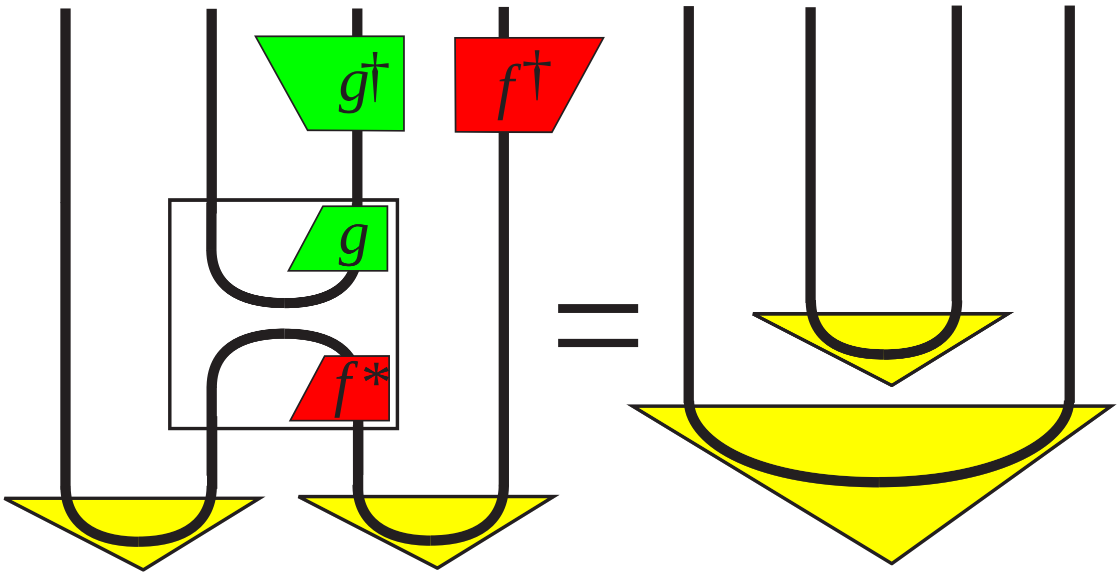
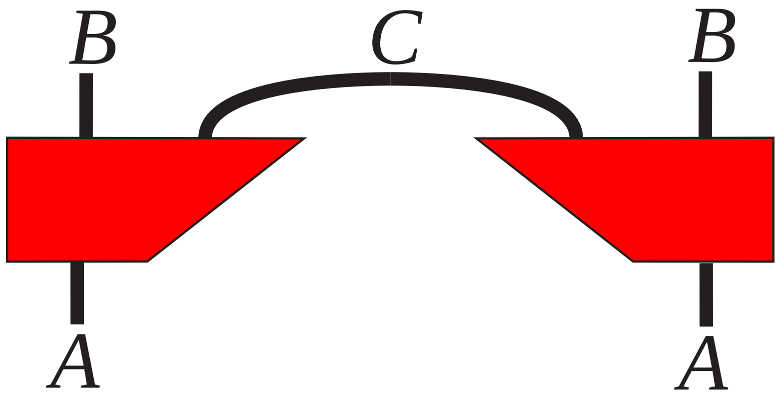


QUANTUM MIXEDNESS

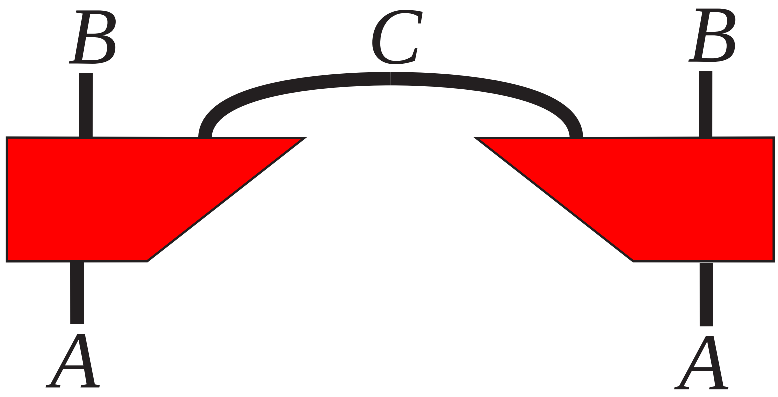
(Selinger 2005)

Construction of mixed states and CPMs



(incarnates Stinespring theorem)

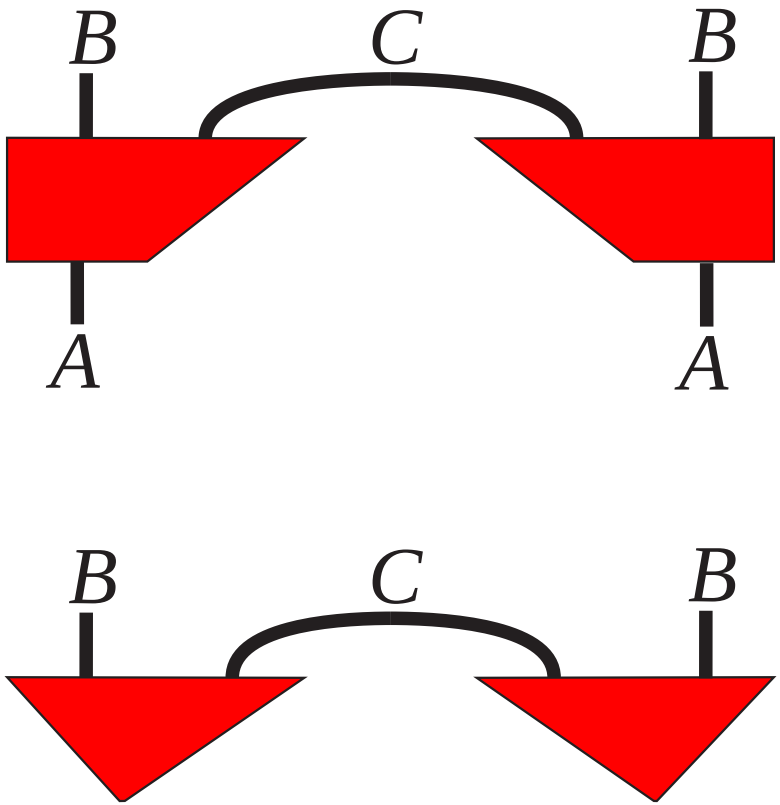
Construction of mixed states and CPMs



Proposition: SM †-structure carries over.

Thm.: Quantum structure carries over.

Construction of mixed states and CPMs



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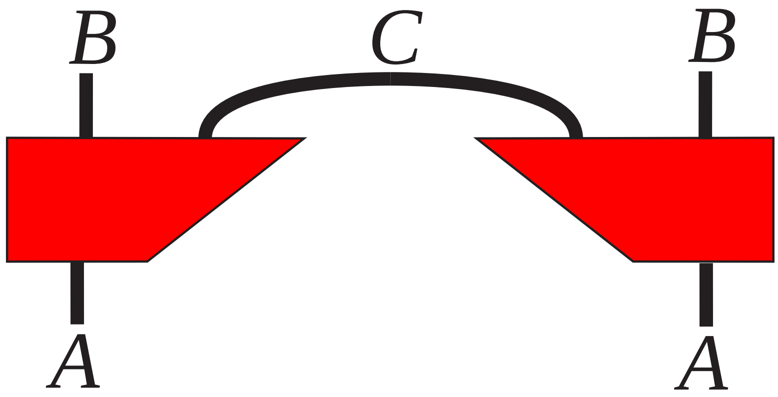
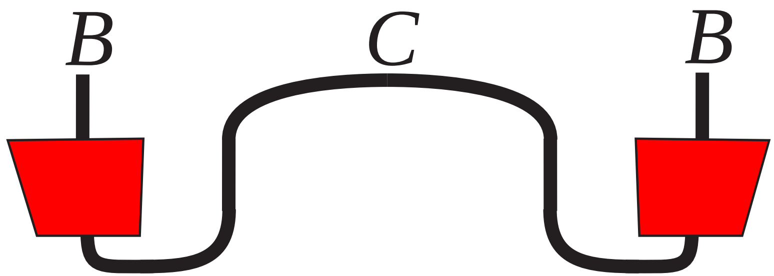
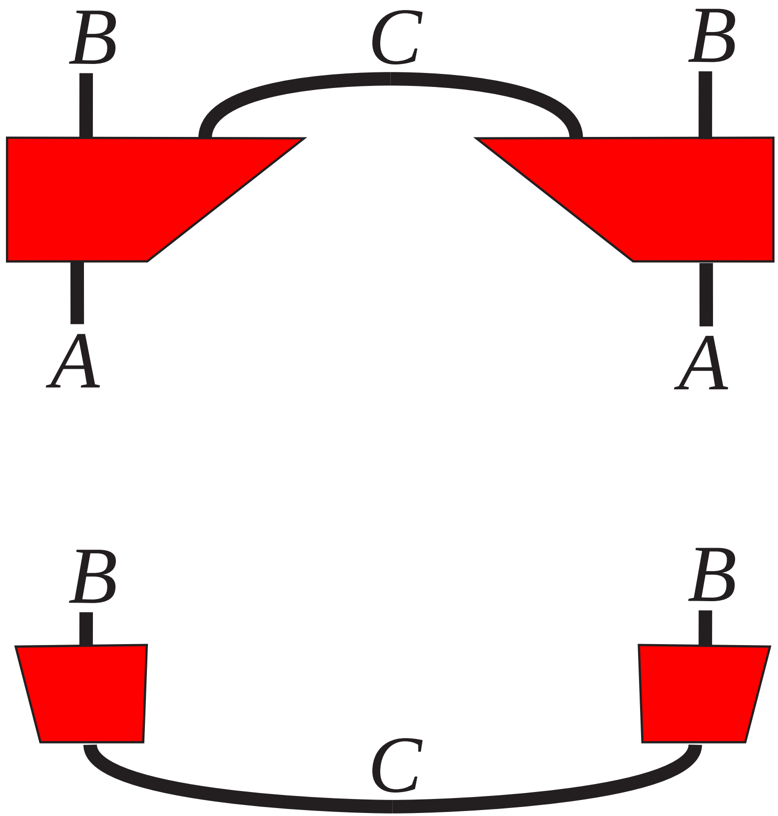
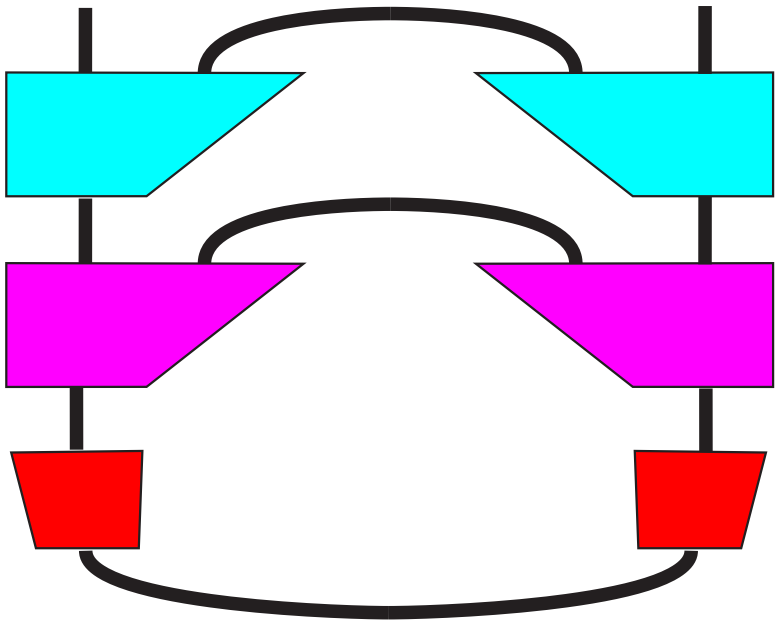




Construction of mixed states and CPMs



Composition of mixed states and CPMs



CLASSICAL STRUCTURE

(Coecke-Paquette-Pavlovic 2006)

Copying ?

Quantum Information obeys a No-Cloning theorem.

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In Linear Logic we drop:

$$\frac{A, B, B \vdash C}{A, B \vdash C} \qquad \text{and} \qquad \frac{A \vdash C}{A, B \vdash C}$$

which is modelled in a $*$-autonomous category.

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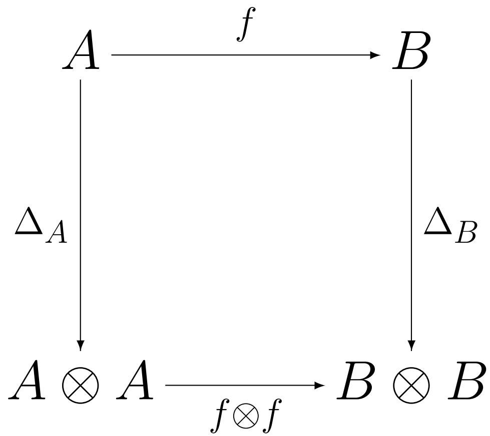
Our $\dagger$-compactness specialises this semantics, and yields No-Cloning and No-Deleting Theorems.

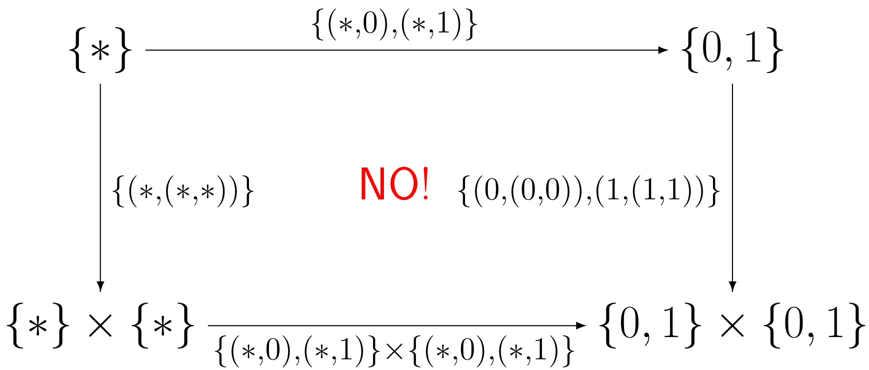
Copying ?

$$\{\Delta_A : A \longrightarrow A \otimes A\}_A$$



No-copying in $(\mathbf{Rel}, \times)$

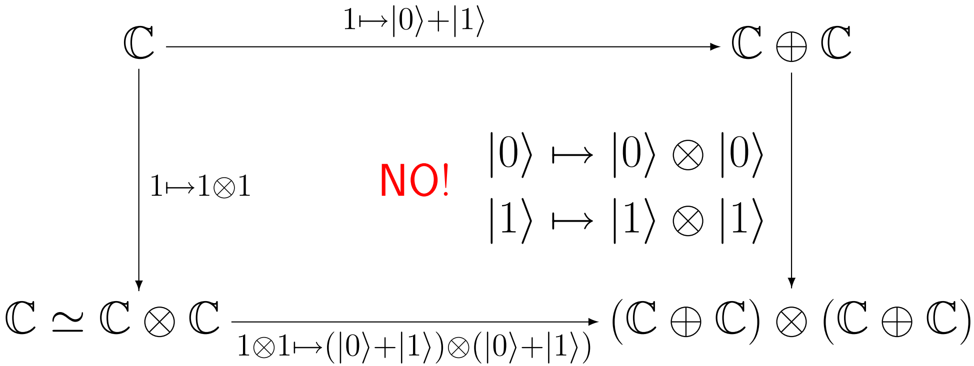
$$\{\Delta_X : x \mapsto (x, x)\}_X$$



$$\{(0,0),(1,1)\} \neq \{0,1\} \times \{0,1\}$$

No-copying of quantum states

$$\{\Delta_{\mathcal{H}} : |\, i\,\rangle \mapsto |\, i\,\rangle \otimes |\, i\,\rangle\}_{\mathcal{H}}$$



No-copying of quantum states

$$\{\Delta_{\mathcal{H}} : |\, i\,\rangle \mapsto |\, i\,\rangle \otimes |\, i\,\rangle\}_{\mathcal{H}}$$

$$\begin{array}{ccc}
\mathbb{C} & \xrightarrow{1 \mapsto |0\rangle + |1\rangle} & \mathbb{C} \oplus \mathbb{C} \\
\downarrow {\scriptstyle 1 \mapsto 1 \otimes 1} & \textsf{NO!} \quad \begin{array}{l} |0\rangle \mapsto |0\rangle \otimes |0\rangle \\ |1\rangle \mapsto |1\rangle \otimes |1\rangle \end{array} & \downarrow \\
\mathbb{C} \simeq \mathbb{C} \otimes \mathbb{C} & \xrightarrow[1 \otimes 1 \mapsto (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)]{} & (\mathbb{C} \oplus \mathbb{C}) \otimes (\mathbb{C} \oplus \mathbb{C})
\end{array}$$

$$|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle \neq (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)$$

Bell-states cause trouble!

Object with classical structure

A commutative comonoid

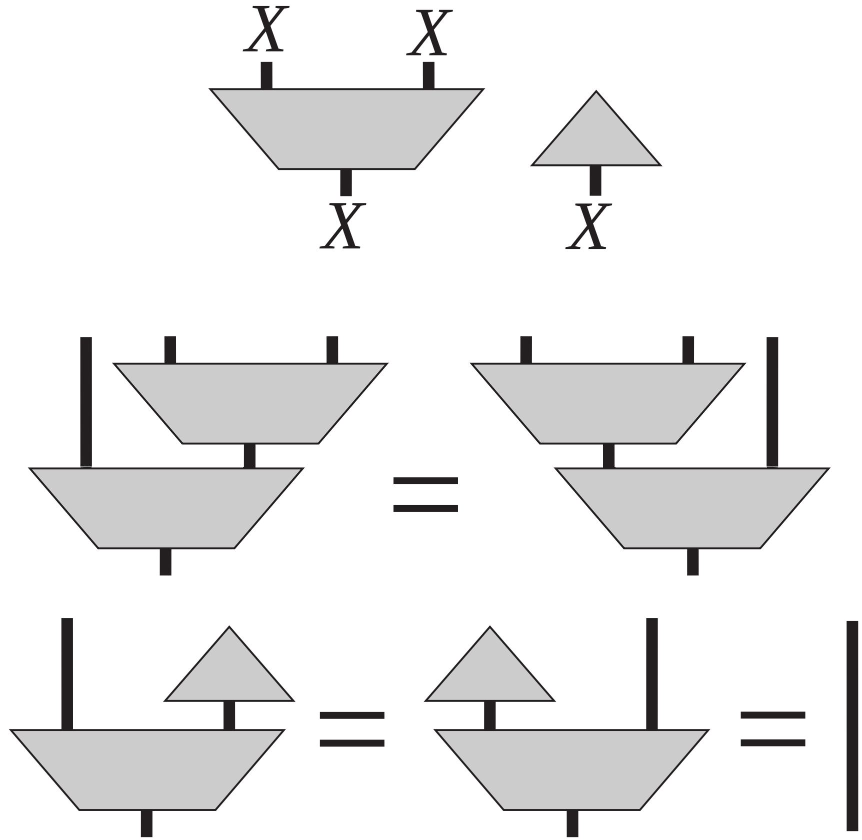
$$(X\,, \delta : X \to X \otimes X\,, \epsilon : X \to \mathrm{I})$$

such that

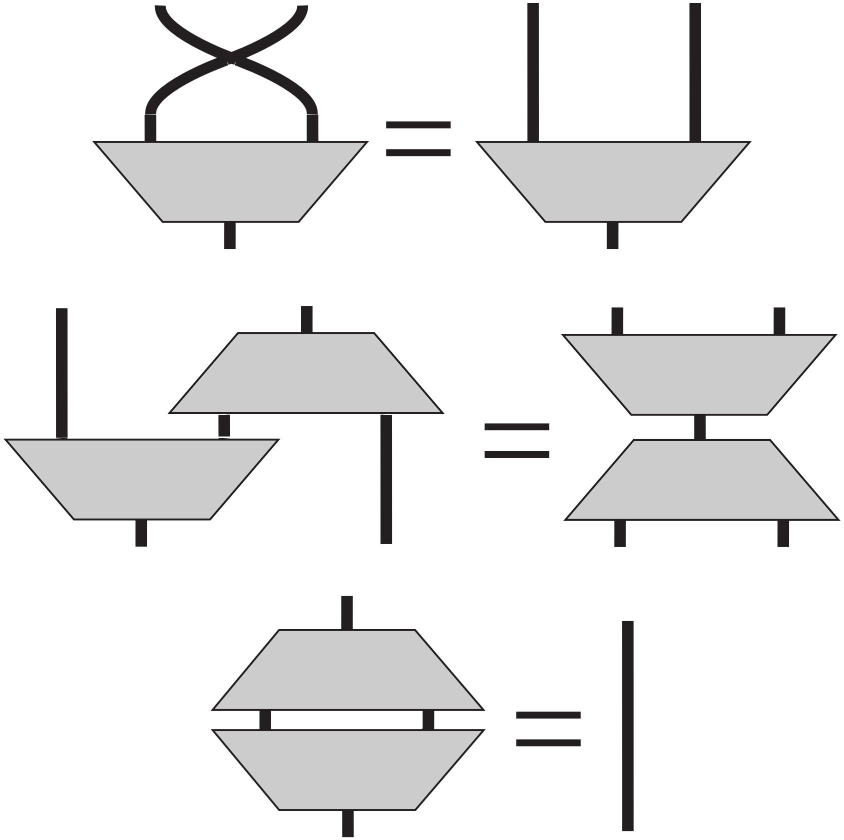
$$\begin{array}{ccc} X \otimes X & \xrightarrow{\ \delta^\dagger\ } & X \\ {\scriptstyle \delta\otimes 1_X}\downarrow & & \downarrow{\scriptstyle \delta} \\ X \otimes X \otimes X & \xrightarrow[\ 1_X\otimes\delta^\dagger\ ]{} & X \otimes X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\ \delta\ } & X \otimes X \\ & {\scriptstyle 1_X}\searrow & \downarrow{\scriptstyle \delta^\dagger} \\ & & X \end{array}$$

commutes. (†-Frobenius & speciality)

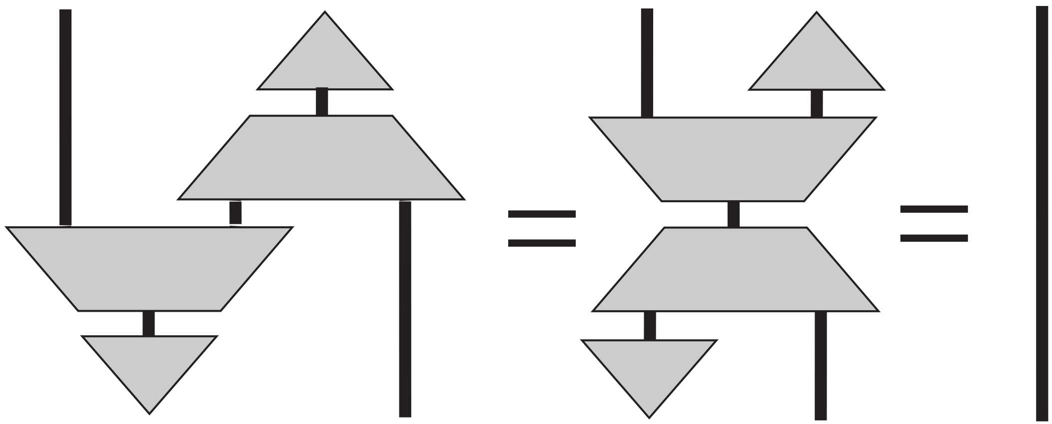
Object with classical structure



Object with classical structure



Classical structure ⇒ quantum structure



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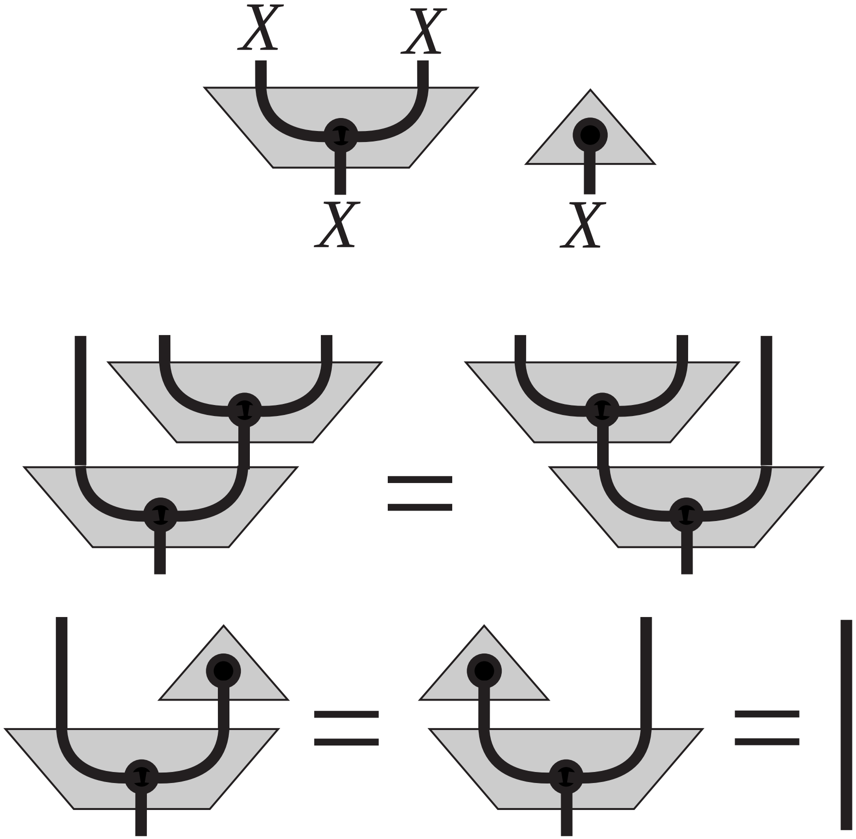

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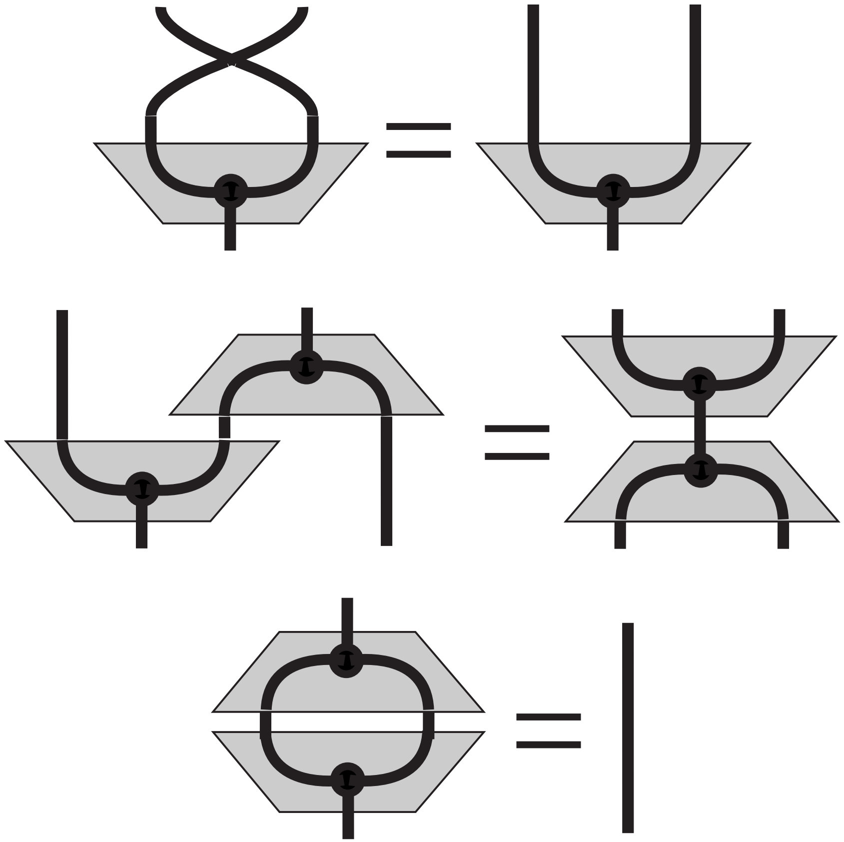
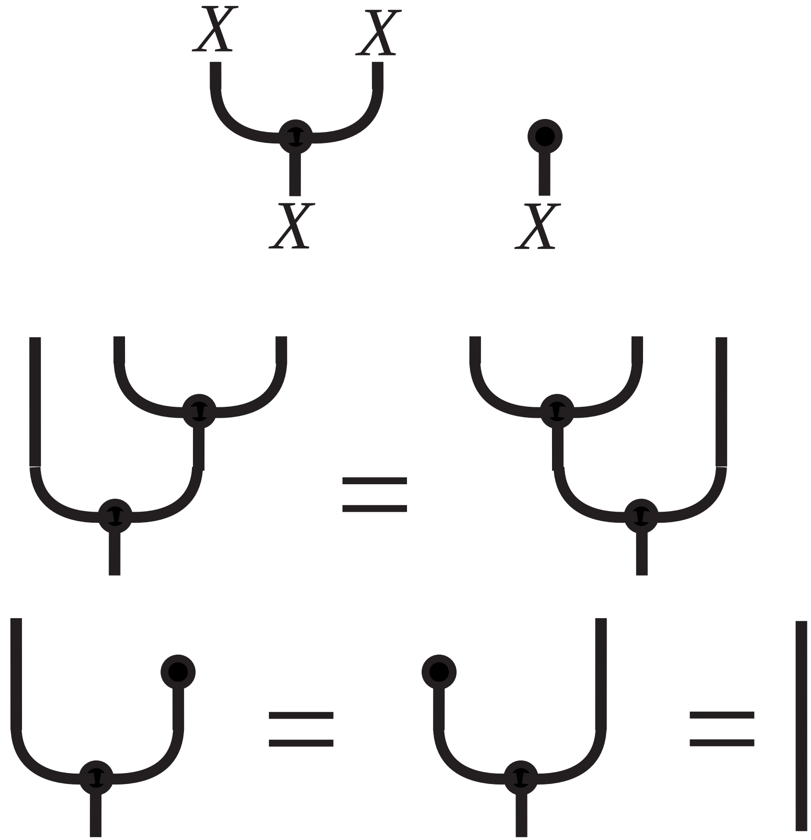
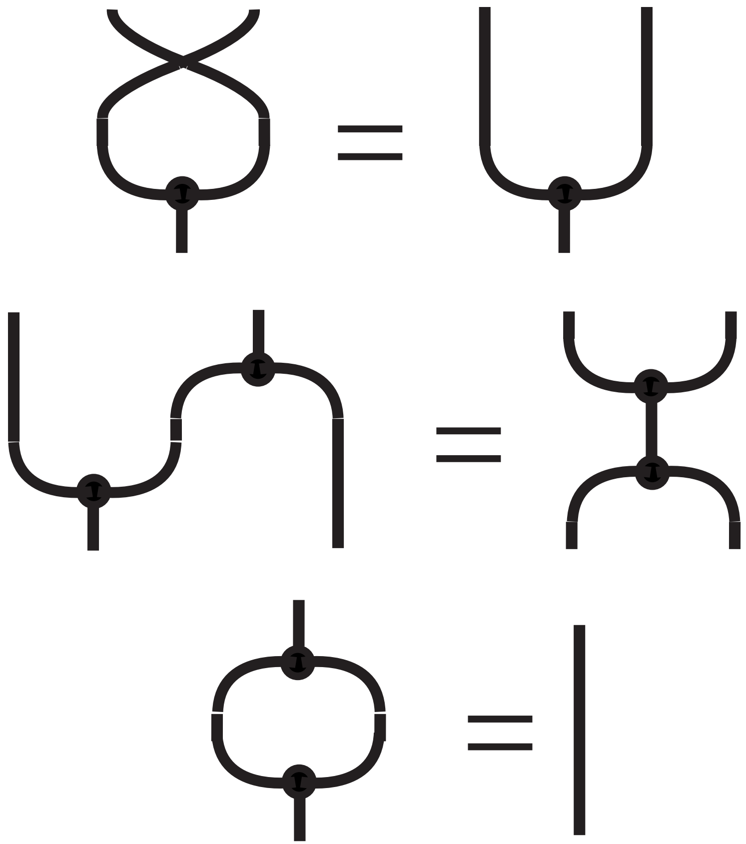


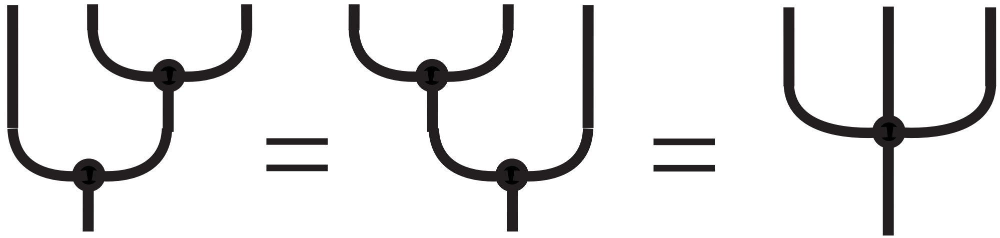
Object with classical structure



Object with classical structure



Notational convention 1:



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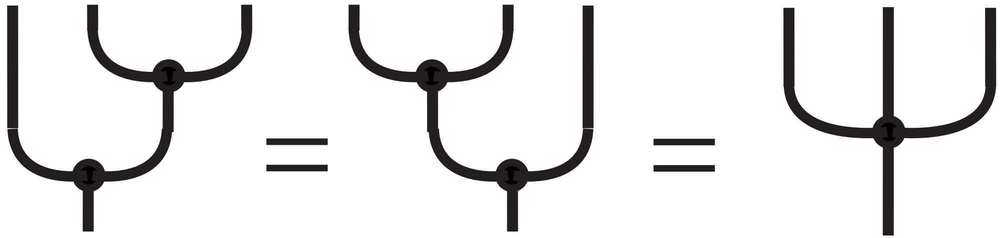
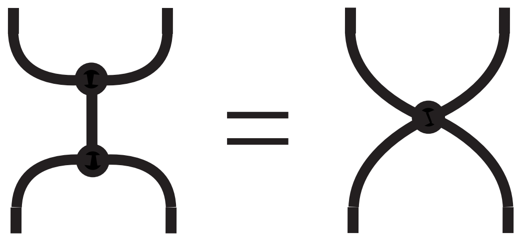


Notational convention 2:



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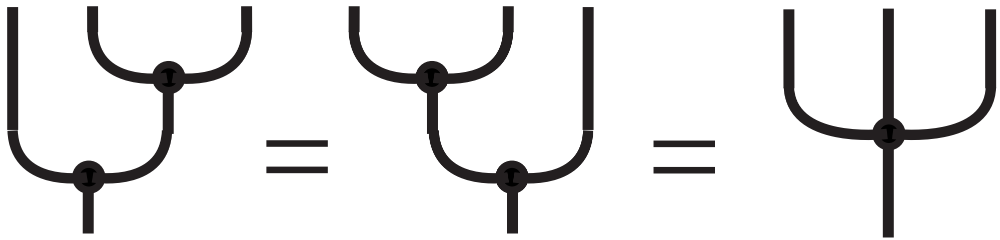
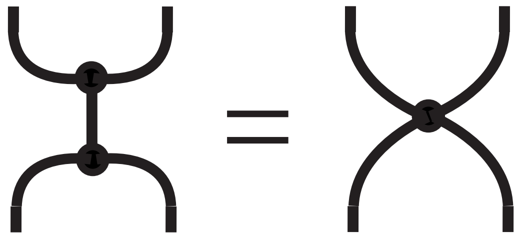
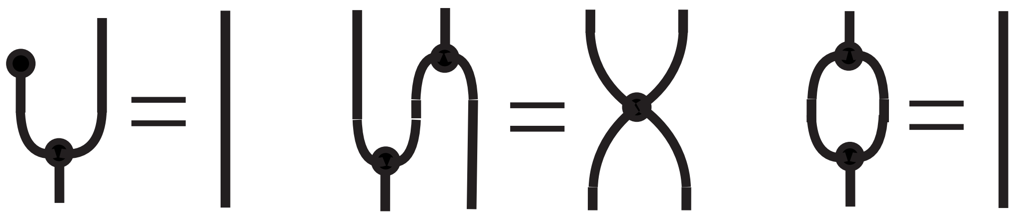


Notational convention 2:

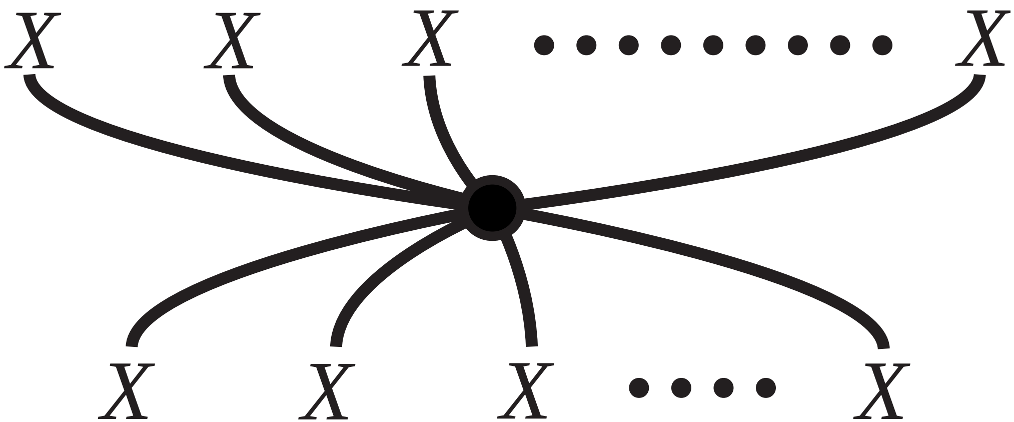


“Fusion” of dots:



"Clean" normalization theorem

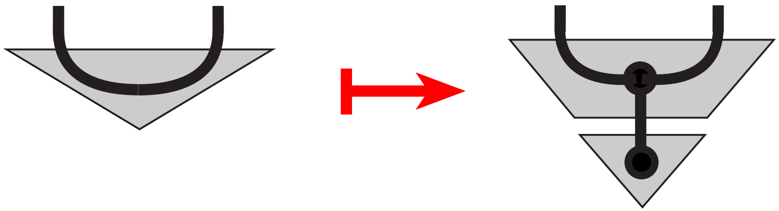
Each "connected" network consisting of δ, $\delta^\dagger$, ϵ, $\epsilon^\dagger$ admits the following normal form through fusion:



(fusions $\sim$ graphical normalising rewriting system)

“Clean” normalization theorem

1.



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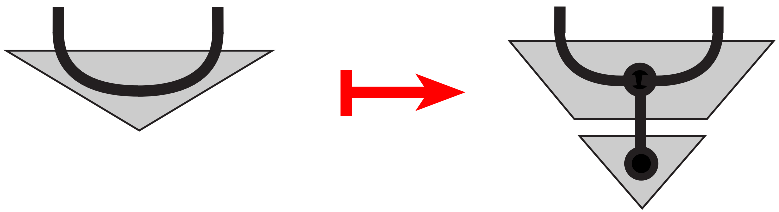
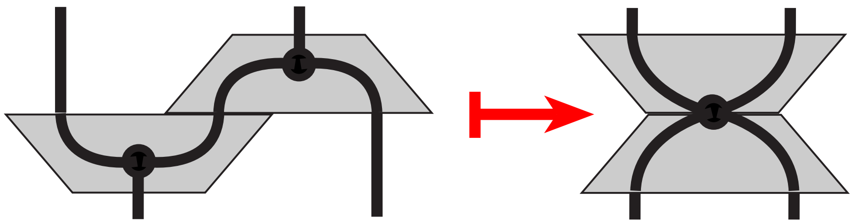


2.



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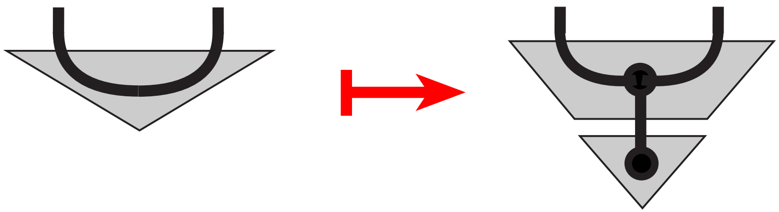

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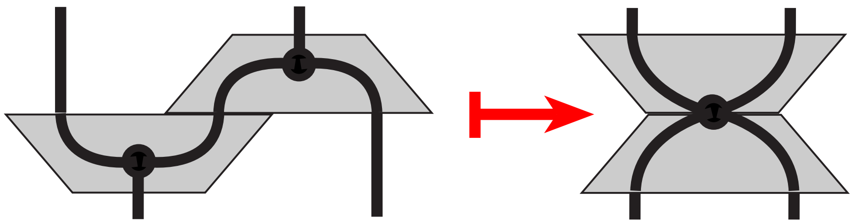
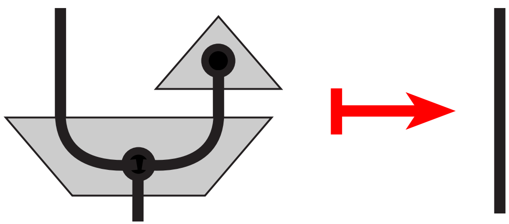


3.



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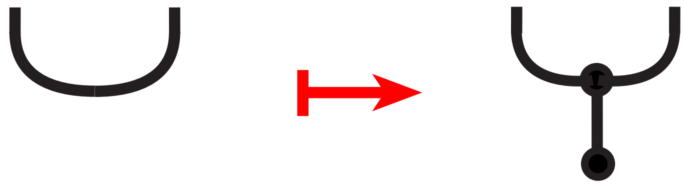

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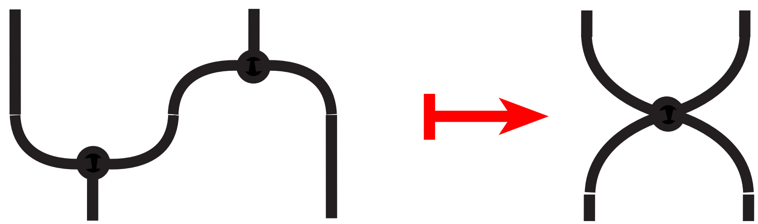

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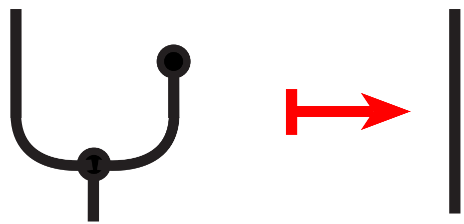
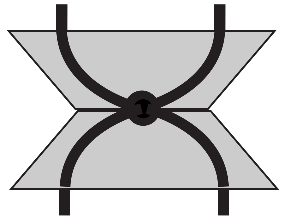


CLASSICAL STOCHASTICITY FROM CLASSICAL STRUCTURE

Diagonal structure on X is

$$\Xi_X := \delta_X \circ \delta_X^\dagger : X \otimes X \to X \otimes X$$



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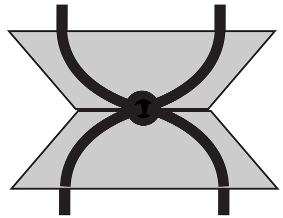

$f : X \otimes X \to Y \otimes Y$ is *diagonal* if

$$f = f \circ \Xi_X = \Xi_Y \circ f$$

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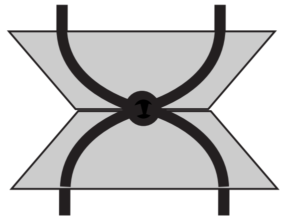


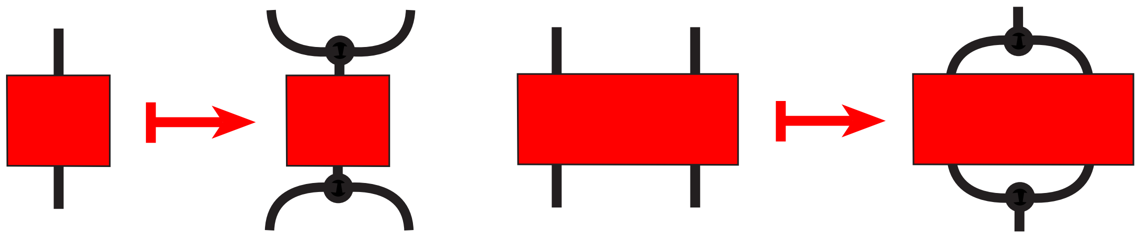
$f : X \otimes X \to Y \otimes Y$ is *diagonal* if

$$f = f \circ \Xi_X = \Xi_Y \circ f$$

Define new category $\mathrm{D}(\mathbf{C})$ with same objects and

$$\mathrm{D}(\mathbf{C})(X, Y) := \{f \in \mathbf{C}(X \otimes X, Y \otimes Y) \mid f \text{ diagonal}\}$$

$$E : \mathbf{C} \longrightarrow \mathrm{D}(\mathbf{C}) :: f \longmapsto \delta_Y \circ f \circ \delta_X^\dagger$$
$$R : \mathrm{D}(\mathbf{C}) \longrightarrow \mathbf{C} :: g \longmapsto \delta_Y^\dagger \circ g \circ \delta_X$$



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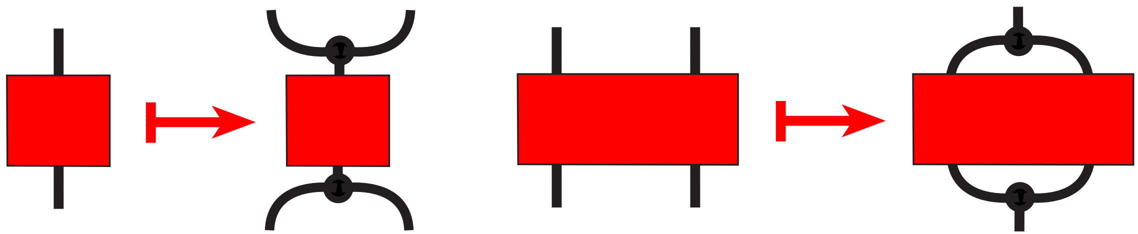

Proposition. E and R are functors.

$$E : \mathbf{C} \longrightarrow \mathrm{D}(\mathbf{C}) :: f \mapsto \delta_Y \circ f \circ \delta_X^\dagger$$

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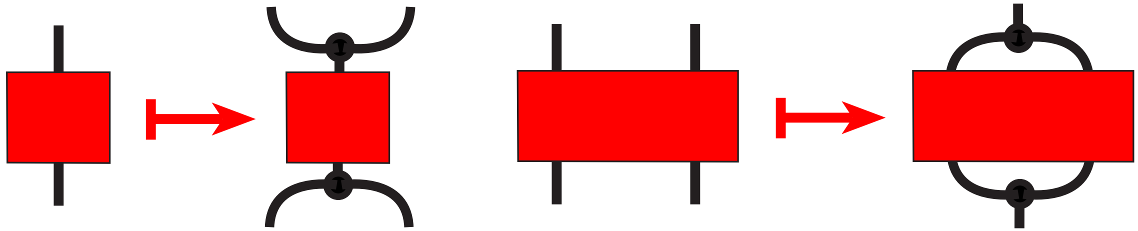
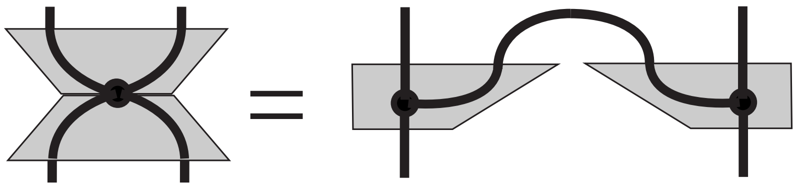


Proposition. E and R are functors.

Proposition. E and R realize isomorphism.

$$\mathbf{C} \underset{R}{\overset{E}{\rightleftarrows}} \mathrm{D}(\mathbf{C}) \qquad (\simeq)$$

Lemma. Diagonal structure is completely positive.



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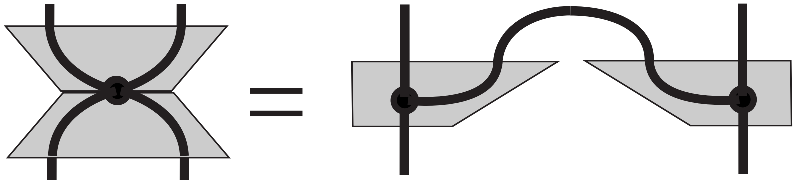

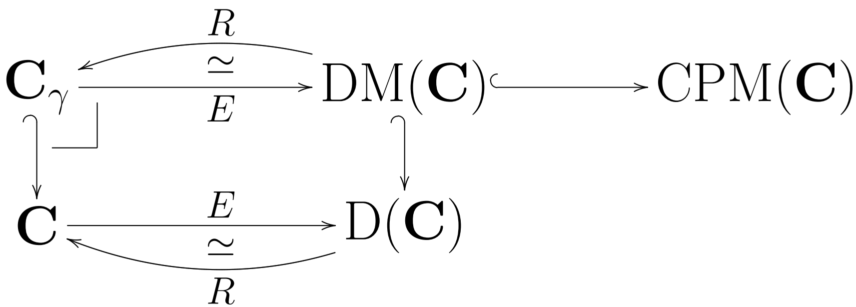

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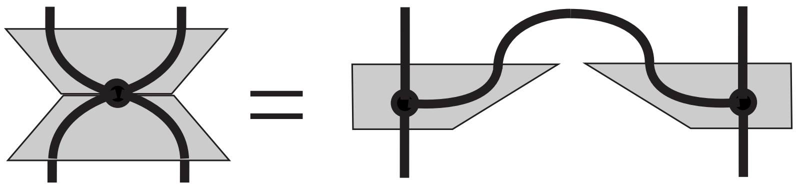
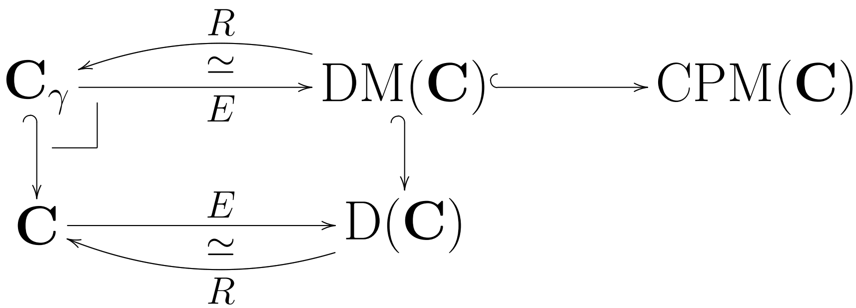
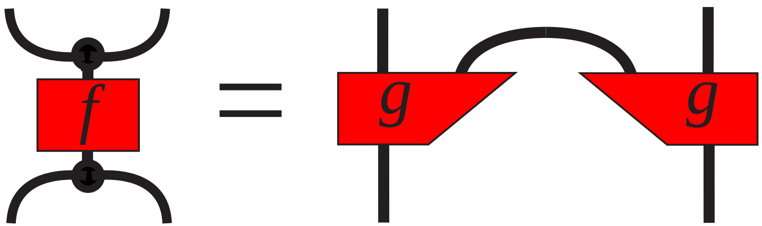


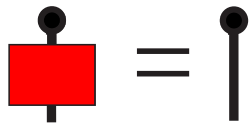




Definition. We call $\mathbf{C}_\gamma \simeq \mathrm{DM}(\mathbf{C})$ the *classical (probability) theory* underlying the quantum theory $\mathbf{C}$.

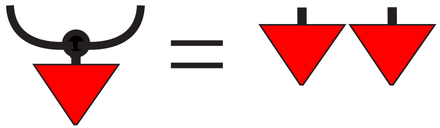
We call the morphisms of $\mathbf{C}_\gamma$ *classical maps*. A classical map is a *stochastic map* if it preserves ϵ i.e.

$$\epsilon_B \circ f = \epsilon_A$$



A stochastic map of type $p : \mathrm{I} \to A$ is a *classical (stochastic) state*. It is a *pure* if it preserves δ i.e.

$$\delta_A \circ p = (p \otimes p) \circ \lambda_\mathrm{I}$$



and $\epsilon_A^\dagger : \mathrm{I} \to A$ is a *maximally mixed state*.

Theorem.

Classical theories underlying quantum theories:

... carry no phase information i.e. $f_* = f$.

... inherit SM †-structure carries over.

... inherit classical structure.

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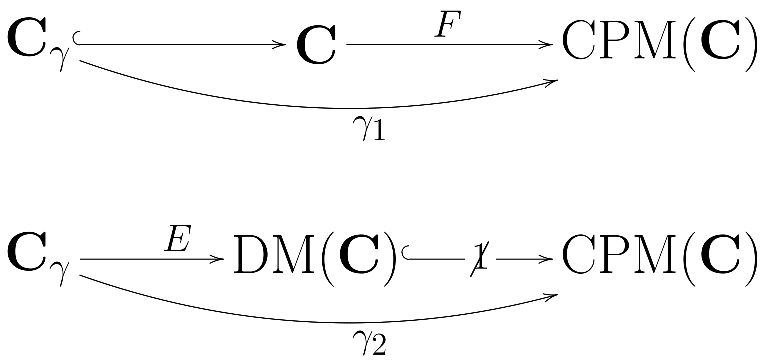
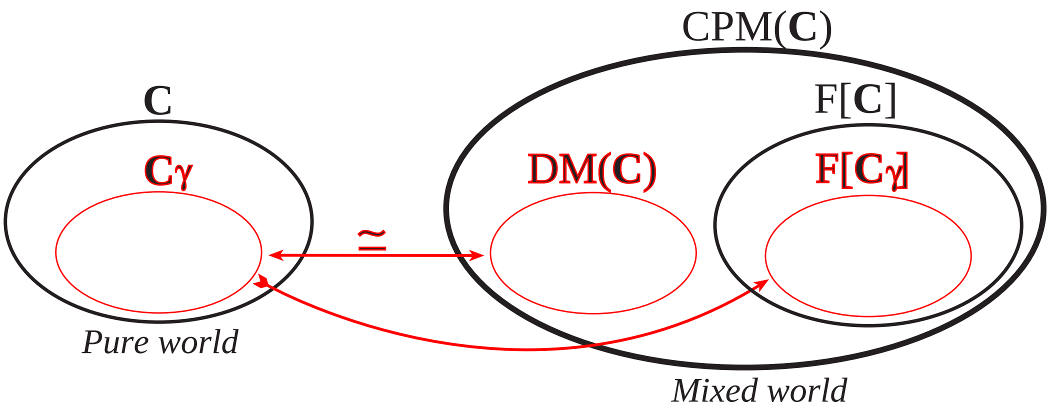
Corollary

No-cloning/No-deleting for classical theories.

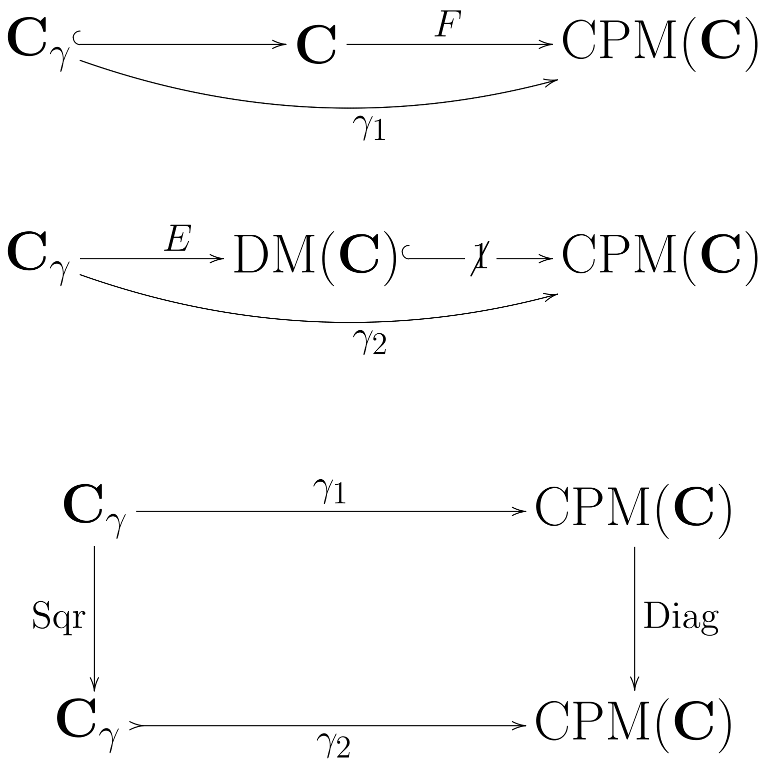
CLASSICAL STOCHASTICITY WITHIN QUANTUM THEORY



$\mathbf{C}_\gamma$
$\mathbf{C}$
F
$\mathrm{CPM}(\mathbf{C})$
γ_1
$\mathbf{C}_\gamma$
E
$\mathrm{DM}(\mathbf{C})$
$\mathrm{CPM}(\mathbf{C})$
γ_2



CPM(**C**)
C
F[**C**]
Cγ
DM(**C**)
F[**C**γ]
≃
Pure world
Mixed world



$\mathbf{C}_\gamma$
$\mathbf{C}$
F
$\mathrm{CPM}(\mathbf{C})$
γ_1
$\mathbf{C}_\gamma$
E
$\mathrm{DM}(\mathbf{C})$
$\mathrm{CPM}(\mathbf{C})$
γ_2
$\mathbf{C}_\gamma$
γ_1
$\mathrm{CPM}(\mathbf{C})$
Sqr
Diag
$\mathbf{C}_\gamma$
γ_2
$\mathrm{CPM}(\mathbf{C})$

MORE CLASSICAL SPECIES FROM CLASSICAL STRUCTURE

Partial maps:

1. $f_* = f$ and preserve δ i.e. $\delta_Y \circ f = (f \otimes f) \circ \delta_X$

Total maps:

2. also preserve ϵ i.e. $\epsilon_Y \circ f = \epsilon_X$

Permutation:

3. also $f^\dagger$ is total.

Relation:

4. $f = \delta_Y^\dagger \circ (f \otimes f_*) \circ \delta_X$.

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Relation:

4. $f = \delta_Y^\dagger \circ (f \otimes f_*) \circ \delta_X$.

Thm. Relations in **C** constitute a *cartesian bicategory* in Carboni and Walter's sense with local partial order:

$$f \subseteq g \;\Leftrightarrow\; f = \delta_Y^\dagger \circ (f \otimes g) \circ \delta_X$$

$$\delta_Y \circ f \subseteq (f \otimes f) \circ \delta_X \qquad\qquad \epsilon_Y \circ f \subseteq \epsilon_X$$

Partial maps:

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Total maps:

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Permutation:

3. also $f^\dagger$ is total.

Relation:

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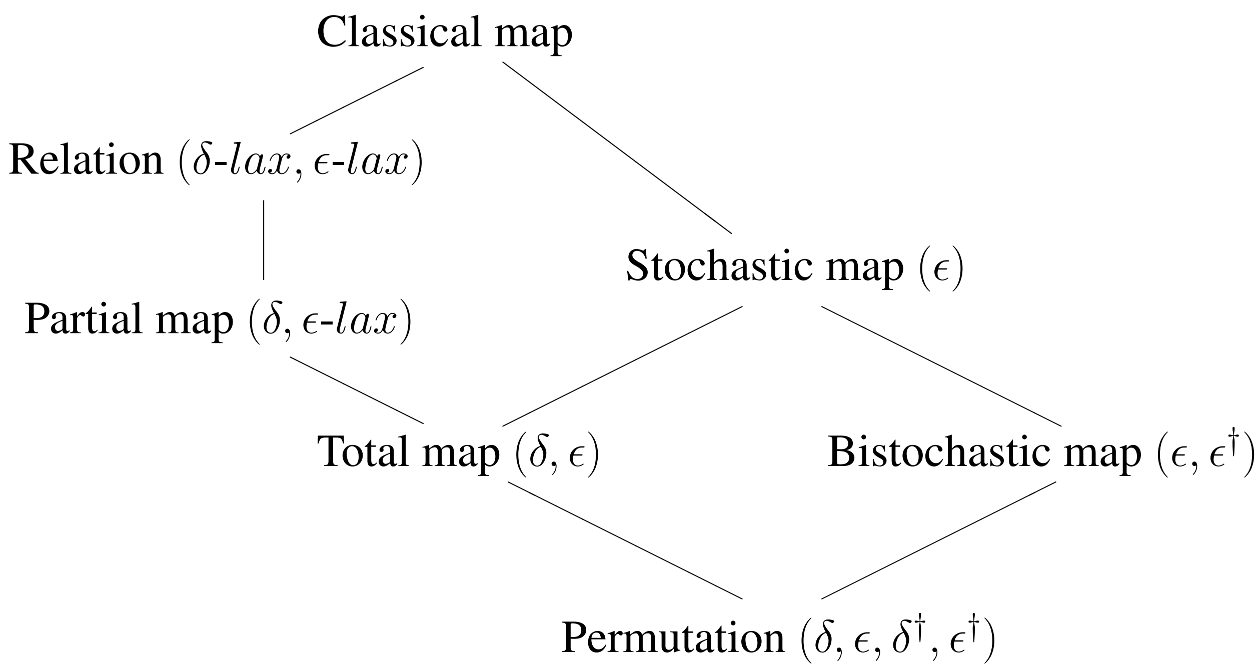
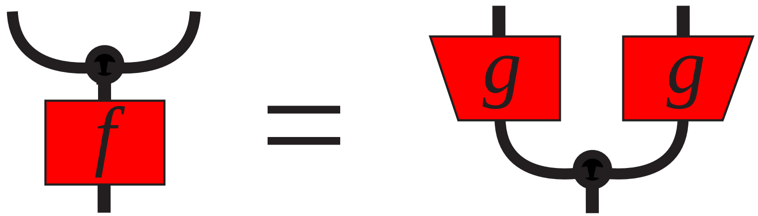
Bistochastic map:

5. both f and $f^\dagger$ are stochastic maps.


Classical map
Relation (δ-lax, ϵ-lax)
Stochastic map (ϵ)
Partial map (δ, ϵ-lax)
Total map (δ, ϵ)
Bistochastic map (ϵ, $\epsilon^\dagger$)
Permutation (δ, ϵ, $\delta^\dagger$, $\epsilon^\dagger$)

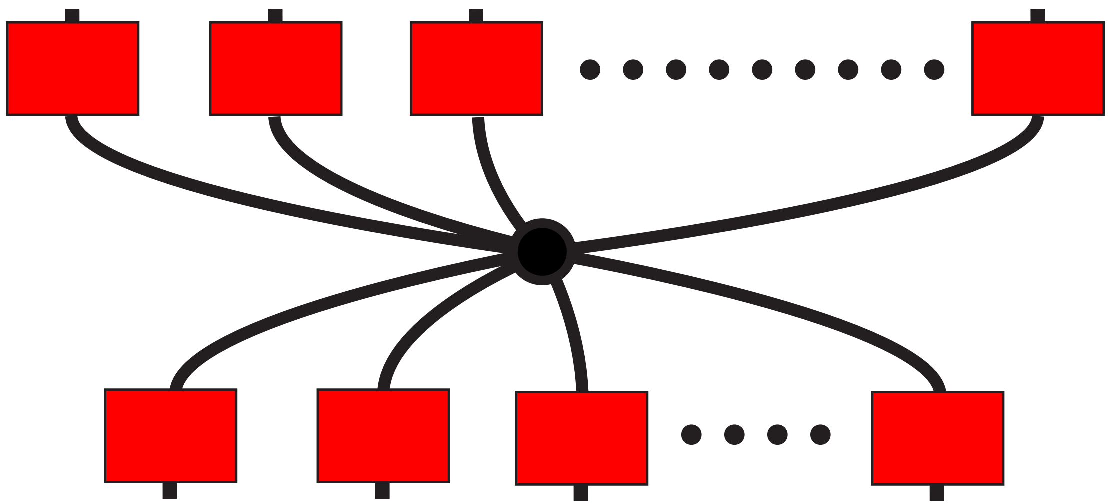
Weighted map:

6. $g : X \to Y$ exist such that $\delta_Y \circ f = (g \otimes g_*) \circ \delta_X$



“Decorated” normalization theorem

Each “connected” network consisting of δ, $\delta^\dagger$, ϵ, $\epsilon^\dagger$ and <u>weighted maps</u> can be rewritten as:



QUANTUM MEASUREMENT FROM CLASSICAL STRUCTURE

Quantum measurement is an operation of type

$$A \to A \otimes X$$

Quantum measurement is an operation of type

$$A \longrightarrow A \otimes X$$

Def. A *quantum measurement* with *outcome spectrum* X is an Eilenbergh-Moore †-coalgebra for $(X \otimes -)$.

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Thm. In **FdHilb** quantum measurement yields usual notion in terms of self-adjoint operators.

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Thm. We can define POVMs and derive Naimarks's.

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Thm. In $\mathbf{FdHilb}$ quantum measurement yields usual notion in terms of self-adjoint operators.

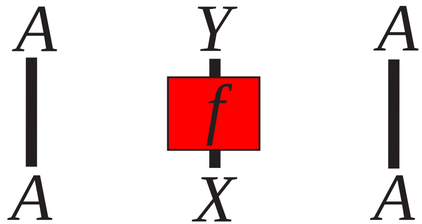
Thm. We can define POVMs and derive Naimarks's.

Control structure and concepts correspond with morphisms in the Kleisli category for $(X \otimes -) : \mathbf{C} \to \mathbf{C}$

CLASSICAL-QUANTUM INTERACTION FROM CLASSICAL STRUCTURE

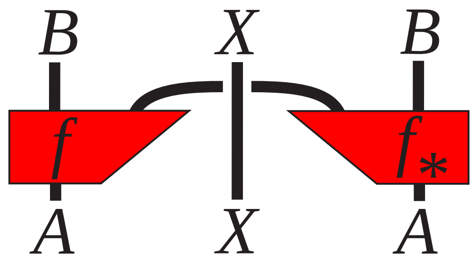
i. *Purely classical* operation if is in the range of

$$\Gamma_A : \mathbf{C}_\gamma \longrightarrow \mathbf{C}_{q+\gamma} :: \begin{cases} X \mapsto (A, X) \\ f \mapsto 1_A \otimes f \otimes 1_A \,. \end{cases}$$



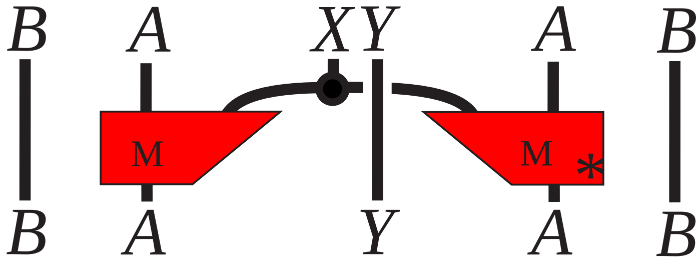
ii. *Purely quantum* operation is in the range of

$$Q_X : \mathrm{CPM}(\mathbf{C}_q) \longrightarrow \mathbf{C}_{q+\gamma} :: \begin{cases} A \mapsto (A, X) \\ f \mapsto \sigma^\dagger \circ (1_X \otimes f) \circ \sigma \,. \end{cases}$$



iii. *Pure measurement is an operation*

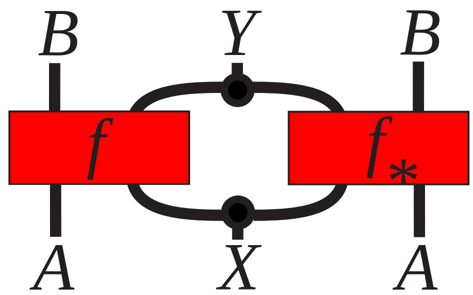
$$\mathcal{M} \in \mathbf{C}_{q+\gamma}(\mathbf{C})((B \otimes A, Y), (B \otimes A, X \otimes Y))$$



iv. Control operations *are co-Kleisli, ...*

v. *Operation is* non-mixed *if it is of the form*

$$(1_B \otimes \delta_Y^\dagger \otimes 1_B) \circ (f \otimes f_*) \circ (1_A \otimes \delta_X \otimes 1_A) \in \mathbf{C}_{q+\gamma}$$



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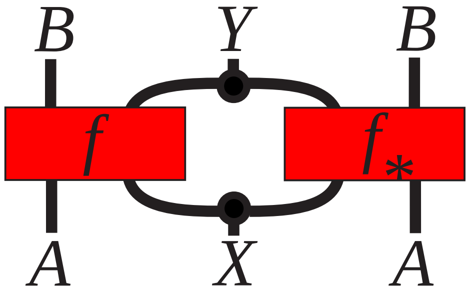

Prop. *A purely classical operation* $\Gamma_A f$ *is non-mixing if and only if* f *is a weighted map.*