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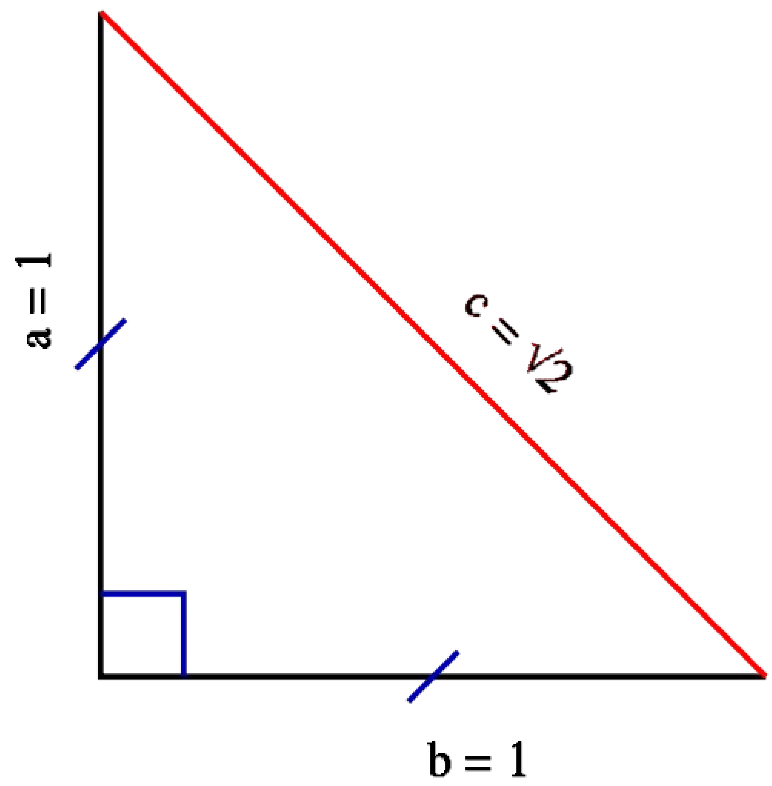
# What is real about the reals

- **Origins**
- **Critique from Physics**
- **Critique from Turing**
- **Chaitin's account**

# Origins

- **Pythagoras theorem**
- **Case of right triangle with sides 1,1, and what?**
- 

# Irrationality of square root 2

**There are many proofs, here is one**

- **Let $\sqrt{2} = p/q$ assume p and q are mutually prime (numbers with no common factors), since we want to express the factors in the simplest way**
- **Their squares are still mutually prime for they are built from the same factors.**
- **Therefore, the fraction $p^2/q^2$ cannot cancel out. In particular, $p^2/q^2$ cannot cancel down to equal 2. Therefore, $p^2/q^2 \neq 2$.**

# Implications

- **If √2 was rational and was equal to p/q this would imply that if we scaled our triangle up, we could have one with sides q,q,p with p and q being integers.**
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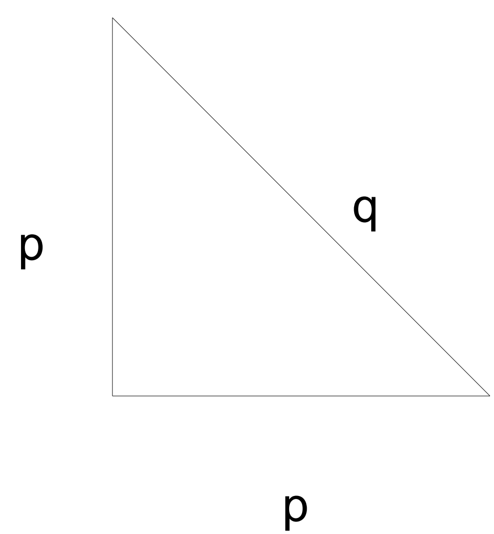



But we have proved that there is no such p,q so that however large we make the triangle, the hypotenuse will never have an integer number of units.

# Infinite divisibility of space?

- This appears to have as a consequence the infinite divisibility of space.
- Suppose we start with a right isoceles triangle with sides 1 meter, 1 meter and third side $\sqrt{2}$
- We then divide the equal sides into p subdivisions – then there will be an integers r >p such that the length of the hypotenuse, in these subdivisions, lies in the range r..r+1.
- for example divide into p=1000 millimeters, then r=1414 millimeters, and hypotenuse is between 1414 and 1415 millimeters.
- Now replace p with r, and recurse. The implication is that space will be infinitely divisible.
- This is the basic intuition or metaphor we have for real numbers.

# Measurement of length

- **Measurement of length requires the use of photons which allow us to measure to an integer accuracy -**
- **The shorter the wavelength of light we use the more accurate our measurement**
- **Hence to measure a hypotenuse more and more accurately we need shorter and shorter wavelengths**

# Energy versus wavelength

$$E = hc/\lambda$$

*Where*

- $E$ *is the energy of the photon,*
- $h$ *is Plancks constant,*
- $\lambda$ *its wavelength,*
- $c$ *speed of light*
- *Using* $E=mc^2$ *we can express this as equivalent mass*

$$m = h/\lambda c$$

# Radius of a black hole

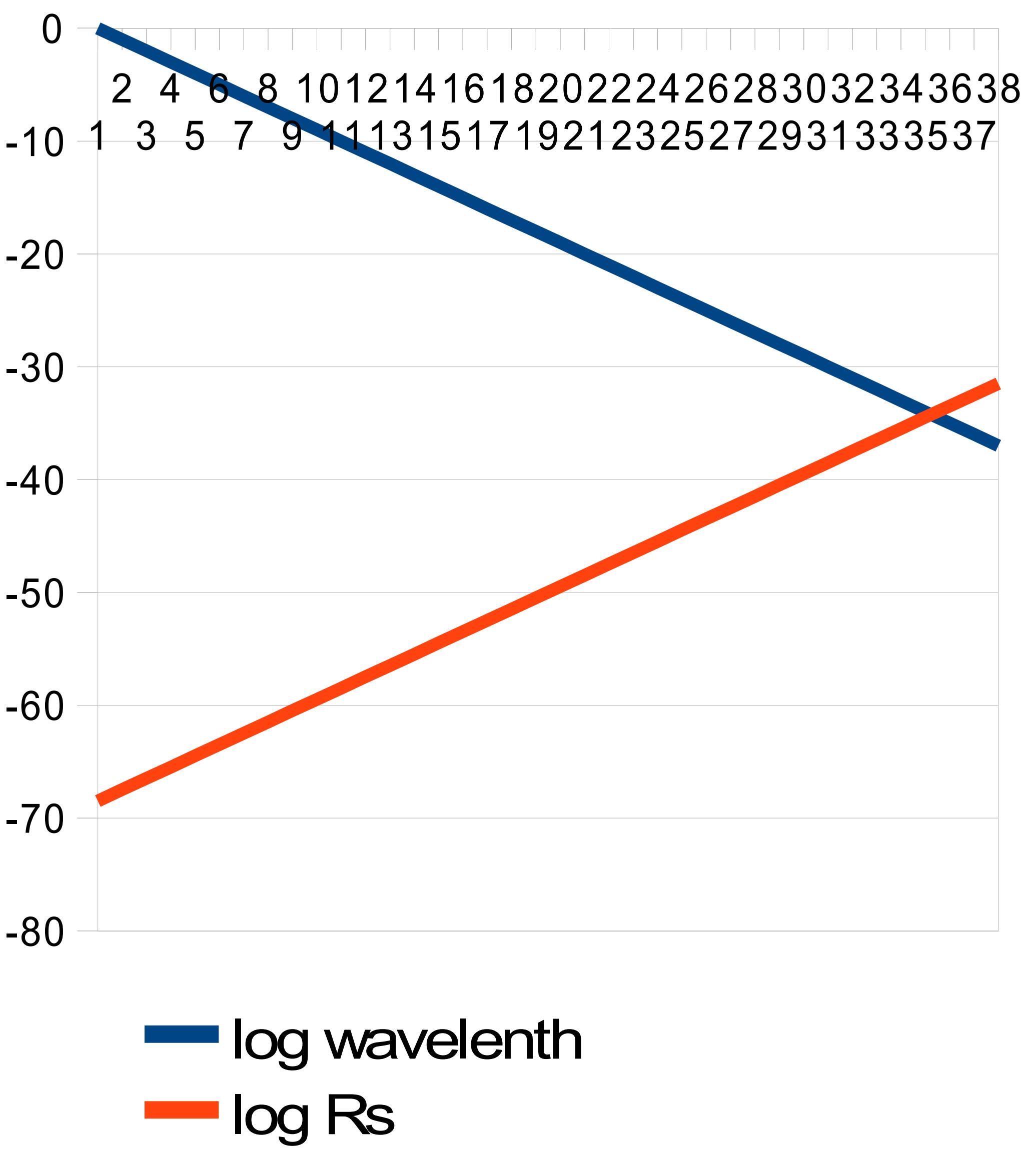
$$r_s = 2Gm/c^2$$

where:

- $r_s$ is the Schwarzschild radius;
- $G$ is the gravitational constant;
- $m$ is the mass of the gravitating object;
- $c$ is the speed of light in vacuum.

Note that the radius is proportional to the mass of the hole, not as you might expect to the cube root of the mass.

# Planck distance




- If we plot on a log scale the black hole radius of the energy of a photon against the wavelength of a photon, we get a crossover at about

  $1.616\ 252 \times 110^{-35}$ meters which is the Planck length
- This is the smallest distance allowed by a combination of relativity and quantum mechanics
- This implies that space is not infinitely divisible
- Generally accepted by all theories of quantum gravity.

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# Not Euclidean geometry

- The infinitely divisible space of Euclidean geometry is thus not 'real'.
- The theorem of Pythagoras applies in the model of space assumed by Euclidean geometry, but is not well defined at very small scales around the Planck length.
- In particular Euclid's assumption of points with position but no magnitude is not well defined in physical reality.
- The geometrical metaphor we are taught of the 'real number line' is thus mistaken.

# Computable reals

- **Turing defined a computable real as a number whose decimal expansion can be computed to any degree of precision by a finite algorithm.**
- **If R is a computable real, there is a function R(n) which when given an integer n will return the nth digit of the number.**

# Implications

- **Any computer programme can be considered as a binary integer, made up of the sequence of bytes of its machine code.**
- **Most such integers are not valid programmes for computing real numbers.**
- **In maths we assume**

$$\text{Reals} \supset \text{Rationals} \supset \text{Integers}$$

- **From a computer science perspective, on the contrary**

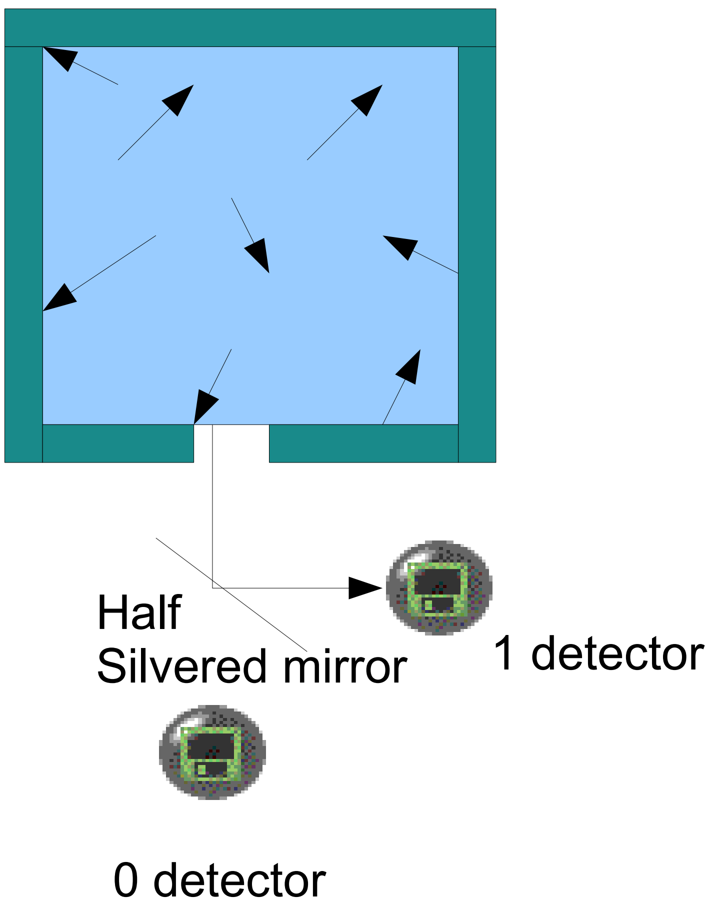
$$\text{Integers} \supset \text{valid Programmes} \supset \text{computable reals}$$

- **A computable real has an encoding that is shorter than its output.**
- **Suppose Length(R(n))=k bits**
- **By setting n>k we can compute more bits of the real than the programme itself contains**
- **Chaitin defines a random bit sequence of length n as one for which no program of length <n exists that will print it out.**
- **Hence any non-computable real is random.**

# How many random reals

- **How many random reals are there?**

**0**




- **A cavity at above background temperature can act as a random bit source**
- **But in providing bits it cools and provides only a finite number of bits**
- **Thus any finite thermal system can only release a finite amount of information and thus only a finite leading bit sequence of a random real – a random integer not a random real.**
- **Note that what we have is the release of information as subsystem moves to thermal equilibrium**

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# Extend to the universe

- **Finite volume of space within our event horizon since big bang**
- **Partition into above average and below average temperature regions**
- **Finite information transfer between these partitions as universe moves to thermal equilibrium**
- **This information constitutes a very long random integer, but it is not yet a random real.**
- **But we started with the whole universe, so there are no random reals anywhere – only finite integers.**