Cantor diagonalisation and planning

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Abstract

Murphy (2006) recently argued that one could use the diagonal argument of the number theorist Cantor to elucidate issues that arose in the socialist calculation debate of the 1930s. We will here argue that Murphy's argument has certain problems, both at the number theoretic level and from the standpoint of economic realism.

1 Is there an infinite number of prices?

Murphy sumarises his argument as follows:

> ...if the socialist planners really are to mimic the market outcome, they would need to publish a list containing, not merely a huge number of prices, and not merely an infinite number of prices, but rather a list containing an uncountably infinite number of prices. But as we have seen above, it is literally impossible, even in principle, for socialist planners to publish such a list. That is, even if we granted them a sheet of paper infinitely long and gave them an infinite amount of time, they still could not, even in theory, write down the entire set of "accounting prices" at which their managers would be required to exchange factors of production. Therefore the purported mathematical solution to Misess challenge is truly impossible to implement, in every sense of the word.

Why is the list of accounting prices that are needed infinite?

Because Murphy, says: "all conceivable goods and services that might be offered, must have corresponding prices included in the planners official lists". This, he contends, includes goods that have not yet been produced - like weekend trips to Mars that may become possible with some future technology. The set of goods which would have to be include, would, he says have to include every possible book that might be written in the future. On this basis he claims that Hayek (1955) grossly underestimated how many equations would actually be required to implement the mathematical solution to the planning problem.

Since computation over infinite domains is in principle impossible, he concludes that the preparation of a socialist plan is not merely intractable, but uncomputable in principle.

Arguments about computability in economics are tricky. At times they reveal more about the axiomatic foundations of economic theories than they do about the operation of real world economies. Arrow and Debreu (1954), for example, supposedly established the existence of equilibria for competitive economies, but as Velupillai (2003) showed, their proof rested on theorems that are only valid in non-constructive mathematics.

Why does it matter whether Arrow used constructive or non-constructive mathematics?

Because only constructive mathematics has an algorithmic implementation and is guaranteed to be effectively computable. But even if

1. a mechanical economic equilibrium can be proven to exist,
2. it can be shown that there is an effective procedure by which this can be determined : i.e., the equilibrium is in principle computable,

there is still the question of its computation tractability. What complexity order governs the computation process that arrives at the solution?

Suppose that an equilibrium exists, but that all algorithms to search for it are NP-hard, that is, the algorithms may have a running time that is exponential in the size of the problem. This is just what has been shown by Deng and Huang (2006). Their result might at first seem to support the Austrian school of economic's contention that the problem of rational economic planning is computationally intractable. In Hayek's day, the notion of NP-hardness had not been invented, but he would seem to have been retrospectively vindicated. Problems with a computational cost that grows as $\mathbf{O}e^n$ soon become astronomically difficult to solve.

We mean astronomical in a literal sense. One can readily specify an NP-hard problem that involves searching more possibilities than there are atoms in the universe before arriving at a definite answer. Such problems, although in principle finite, are beyond any practical solution.

But this knife cuts with two edges. On the one hand it shows that no planning computer could solve the neo-classical problem of economic equilibrium. On the other it shows that no collection of millions of individuals interacting via the market could solve it either. In neo-classical economics, the number of constraints on the equilibrium will be proportional, among other things, to the number of economic actors n. The computational resource constituted by the actors will be proportional to n but the cost of the computation will grow as e^n. Computational resources grow linearly, because they are proportional to the number of people available to make decisions, computational costs grow exponentially. This means that a market economy could never have sufficient computational resources to find its own mechanical equilibrium.

Should we conclude from this that market economies are impossible?

Clearly not as we have empirical evidence that they exist. It follows that the problem of finding the neo-classical equilibrium is a mirage. No planning system could discover it, but nor could the market. The neo-classical problem of equilibrium misrepresents what capitalist economies actually do and also sets an impossible goal for socialist planning.

If you dispense with the notion of mechanical equilibrium and replace it with statistical equilibrium one arrives at a problem that is much more tractable. The simulations of Wright (2003, 2005), Dragulescu and

Yakovenko (2000) show that a market economy can rapidly converge on this sort of equilibrium. But as we have argued above, this is because regulation by the law of value is computationally tractable. This same tractability can be exploited in a socialist planning system. We would contend that economic planning does not have to solve the impossible problem of neo-classical equilibrium, it merely has to apply the classical law of value more efficiently.

Consider Murphy's thesis, he alleges that the problem domain of economic calculation is not merely NP hard, but actually transfinite. If the problem domain is actually infinite, how is a market economy supposed to provide an effective solution. No finite computational resource, whether it be state planners with computers or capitalist supermarkets and wholesalers with their computers and databases, can scan an infinite search space[1]. In that case, either the market must also be deficient, by Murphy's criteria or his criteria are misplaced. Murphy is demanding the impossible, the backward transmission of information through time. He is demanding that an economic system today take into account information which can only exist in the future: information about products that will one day be invented in the future. No system, whether capitalistic or socialistic could do this. Economic systems can only allocate resources between products that have already been thought of or invented.

What is the author's motivation for the apparently bizarre assertion that socialist planners are under an obligation to produce prices for all commodities, present and future?

His idea is that only on this condition can the planners claim that a Lange/Dickinson-type system (patterned after the neoclassical fiction of the Walrasian auctioneer) is a "perfect" substitute for the market mechanism, with regard to the issue of innovation.

There is a small kernel of sense in this, though it is expressed perversely. A persistent theme in Austrian economics is that the neoclassical representation of the market system, with its stress on static allocative efficiency, is misleading and in a sense sells short the virtues of capitalism. The principal virtue of capitalism, according to the Austrians, is not that it produces an optimally efficient, perfectly competitive equilibrium with prices everywhere equal to marginal cost, but that it spawns an effective process of discovery and innovation – the whole "entrepreneurial" thing.

If an economic system were to entrust its process of product- and process-innovation purely and simply to a mechanism in which managers make decisions on what to produce, and how to produce it, based on given relative prices, then in a sense the author is right: the "given relative prices" would have to include the prices of all the things they *might* produce as well as things they're currently producing.

Our response is twofold. First, historically, the Lange/Dickinson scheme was not supposed to be a solution to the problem of innovation: that was not the problem von Mises (1935) originally posed. Second, the market does not handle innovation purely via passive responses to price signals, and by the same token a socialist economy will not handle innovation

[1] We here disregard the highly contentious recent claims of Kieu (2001) for the reasons given in Tsirelson (2001), Smith (2006).

Table 1: Convergence of gross production on that required for the final net product

iron	coal	corn	bread	labour	
0	20000	0	1000	0	Net output
2000	24500	1500	1000	61000	1st estimate gross usage
2580	29400	1650	1000	129500	
3102	31540	1665	1000	157300	
3342	33012	1666	1000	174310	
..	..	..	..	..	hidden steps
3708	34895	1667	1000	196510	
3708	34895	1667	1000	196515	
3708	34896	1667	1000	196517	20th estimate gross usage

via passive responses to computed prices (or labour values) of currently non-existent goods.

In any system, what is needed is some mechanism for exploring options "in the neighbourhood of" the current input-output matrix that are rendered feasible by scientific advances (or, in some cases, just by leaps of the imagination). This inevitably involves experimentation, trial and error, and so on. This task is beyond the scope of the Lange/Dickinson mechanism, just as it is beyond the scope of the "standard" process of market equilibration (migration of capital from low-profit fields to high-profit fields). Creating an effective mechanism for this job is non-trivial. Cottrell and Cockshott (1992) discuss this, suggesting that one would need some kind of agreed annual innovation budget; that it might be a good idea to have more than one agency in the business of disbursing resources for innovation experiments – but the issue could stand more thought. The parlaying of scientific advances into new products that people want, or new processes that are more efficient than the old ones, is not an issue that invites a simple "capitalism vs socialism" split. Capitalist economies have differed quite widely in their effectiveness in this regard, and socialist economies might be expected to differ too.

If we assume that the socialist economy retains some form of market for consumer goods as proposed by Lange to provide information on final requirements then the process of deriving a balanced plan is tractable.

Let us take a very simple example, an economy with 4 types of goods which we will call bread, corn, coal and iron. In order to mine coal, both iron and coal are used as inputs. To make bread we need corn for the flour and coal to bake it. To grow the corn, iron tools and seed corn are required. The making of iron itself demands coal and more iron implements. We can describe this as a set of four processes:

1 ton iron	← 0.05 ton iron + 2 ton coal + 20 days labour
1 ton coal	← 0.2 ton coal + 0.1 ton iron + 3 days labour
1 ton corn	← 0.1 ton corn + 0.02 ton iron + 10 days labour
1 ton bread	← 1.5 ton corn * 0.5 ton coal + 1 days labour

Assume, following Lange (1938), that the planning authorities have a current estimate of consumer demand for final outputs. The planners start with the required net output. This is shown on the first line of Table 1. We assume that 20000 tons of coal and 1000 tons of bread are the consumer goods required.

They estimate how much iron, corn, coal, and labour would be directly consumed in producing the final output: 2000 tons of iron, 1500 tons of corn and 4500 additional tons of coal.

They add the intermediate inputs to the net output to get a first estimate of the gross usage of goods. Since this estimate involved producing more iron, coal and corn than they had at first allowed for, they repeat the calculation to get a second estimate of the gross usage of goods.

The answers differ each time round, but the differences between sucessive answers get smaller and smaller. Eventually, (assuming integer quantities are used) after 20 attempts in this example, the planners get a consistent result: if the population is to consume 20000 tons of coal and 1000 tons of bread, then the gross output of iron must be 3708 tons, coal must be 34896 tons and that that of corn 1667 tons.

Is it feasible to scale this up to the number of goods produced in a real economy?

Whilst the calculations would have been impossibly tedious to do by hand in the 1930s, they are readily automated. Table 1 was produced by running a computer algorithm. If detailed planning is to be feasible, we need to know:

1. How many types of goods an economy produces.
2. How many types of inputs are used to produce each output.
3. How fast a computer program running the algorithm would be for the scale of data provided in (1) and (2).

Table 2 illustrates the effect of running the planning algorithm on a cheap personal computer of 2004 vintage. We determined the calculation time for economies whose number of industries ranged from one thousand to one million. Two different assumptions were tested for the way in which the mean number of inputs used to make a good depends on the complexity of the economy.

It is clear that the number of direct inputs used to manufacture each product is only a tiny fraction of the range of goods produced in an economy. It is also plausible that as industrial complexity develops, the mean number of inputs used to produce each output will also grow, but more slowly. In the first part of Table 2 it is assumed that the mean number of inputs (M) grows as the square root of the number of final outputs (N). In the second part of the table the growth of M is assumed to follow a logarithmic law.

It can be seen that calculation times are modest even for very big economic models. The apparently daunting million equation foe, yields gracefully to the modest home computer. The limiting factor in the experiments is computer memory. The largest model tested required 1.5 Gigabytes of memory larger models would have required a more advanced 64-bit computer.

Table 2: Timings for applying the planning algorithm to model economies of different sizes. Timings were performed on a 3 Ghz Intel Zeon running Linux, with 2 GB of memory.

	Industries N	Mean Inputs M	CPU Time seconds	Memory bytes
Law $M = \sqrt{N}$				
	1,000	30	0.1	150KB
	10,000	100	3.8	5MB
	40,000	200	33.8	64MB
	160,000	400	77.1	512MB
	320,000	600	166.0	1.5G
Law $M \approx \log N$				
	1,000	30	0.1	150KB
	10,000	40	1.6	2.4MB
	100,000	50	5.8	40MB
	1,000,000	60	68.2	480MB

The experiment went up to 1 million products. The number of industrial products in the Soviet economy was estimated by Nove (1983) to be around of 10 million. Nove believed this number was so huge as to rule out any possibility of constructing a balanced disagregated plan. This may well have been true with the computer technology available in the 1970s, but the situation is now quite different.

2 Is there an uncountably infinite number of prices?

Murphy claims to use Cantor's diagonal argument to demonstrate that there is an uncountable infinity of prices. In fact, he does no such thing. Rather, he explains diagonalisation and then asserts that it is applicable to the alleged infinity of prices without actually doing so. Nonetheless, let us, for the sake of arguement, assume that there is an infinite number of prices and explore its cardinality.

Cantor's arguemnt may be summarised briefly as follows. We may enumerate (i.e. list or write down) all the integers starting from one by repeatedly adding one:

```
1
2
3
...
```

We may also enumerate all the rational numbers, that is numbers made from ratios of integers, by systematically listing all possible successive ratios of integers:

```
1/1
```

```
1/2
2/1
2/2
1/3
2/3
3/3
3/2
3/1
...
```

Note that many rationals recur. For example, `1` is `1/1` and `2/2` and `3/3` and so on. Note also that the cardinality of the rationals, that is the "type of infinity" that characterises how many there are, is the same as that of the integers, because we can put the rationals into one to one correspondence with the integers:

```
1 1/1
2 1/2
3 2/1
4 2/2
5 1/3
...
```

In other words, there are as many rationals as integers. We say that the rationals are *countable*.

It is important to note that every integer and rational has a finite representation, even though some rationals have infinte expansions. For example, if we try to evaluate `1/3`, we get `0.33333...` with `3` repeating forever. Nonetheless, `1/3` is a perfectly good finite representation of that value.

Cantor's introduced diagonalisation to show that the number of real numbers, that is numbers consisting of an integer followed by an arbitrary number of decimal places, has a higher cardinality than the integers and rationals. That is there are more reals than integers or rationals. Following Kleene's account Kleene (1952), we consider all the real numbers between `0` and `1` where each is represented uniquely by a decimal fraction that doesn't terminate. Any number whose last decimal digit is `0` has that replaced with an infinite number of `9`s. Now, suppose there is an enumeration of reals x_1 x_2 x_3 ... between `0` and `1`. Suppose x_i has decimal digits x_{i1} x_{i2} x_{i3} and so on. Then we can write down the sequence of decimal fractions as:

$.x_{11}\ x_{12}\ x_{13}\ ...$
$.x_{21}\ x_{22}\ x_{23}\ ...$
$.x_{31}\ x_{32}\ x_{33}\ ...$
$...$

We now construct a new decimal fraction $x^{'}$ such that $x^{'}_{11}$ differs from x_{11}, $x^{'}_{22}$ differs from x_{22}, $x^{'}_{33}$ differs from x_{33}, and so on so that in general $x^{'}_{ii}$ differs from x_{ii}. Thus, $x^{'}$ is different from all of the reals that we have listed between `0` and `1`. We conclude that the cardinality of the reals is higher than that of the integers and rationals; in other words, the reals are not countables.

It is now easy to demonstarte that this argument does not apply to prices. First of all, unit prices are only representable to a finite number of places as monetary systems are based on integer quantities of the smallest values. We might argue that we wish to deal in arbitrary fractions of prices, for example in selling arbitrary proportions of a kilogram of cheese. Ignoring the physical limitations on measurement which ensure that we can only distinguish discrete quantities of things on the microscopic level **?**, every fraction is ratio of integers and so must be rational and therefore countable. Thus any attempt to apply diagonalisation will necessarily produce a value which has been enumerated. Finally, we are not interested in prices per se but in prices of commodities. As the number of different commodities is necessarily countable, so is the number of corresponding prices.

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