A NOTE ON THE USE OF VECTOR SPACE METRICS

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ABSTRACT. It is argued that the vector space measures used to measure closeness of prices and labour values are invalid because of the observed metric of commodity space. An alternative vector space within which such measures do apply is proposed. It is shown that commodity exchange can be modeled by the application of unitary operators to this space.

In the recent literature relating to measuring the closeness of price of production vectors to value vectors [2, 3, 4, 5]it has been taken as given that the use of vector space measures is appropriate. I wish to point out that this is at least questionable.

1. THE VECTOR SPACE PROBLEM

Vector spaces are a subclass of metric space. A metric space is characterized by a positive real valued metric function $\delta(p,q)$ giving the distance between two points, p,q. This distance function must satisfy the triangle inequality $\delta(p,q) \leq \delta(p,r) + \delta(q,r)$. In vector spaces this metric takes the form:

$$\delta(\mathbf{p},\mathbf{q}) = \sqrt{\sum (p_i - q_i)^2} \tag{1.1}$$

We have argued elsewhere[1] that the metric of commodity space does not take this form. Let us recapitulate the argument.

Conjecture 1.1. *Commodity space is a vector space.*

Assume that we have a commodity space made up of two commodities, gold and corn and that 1oz gold exchanges for 100 bushels of corn. We can represent any agent's holding of the two commodities by a 2 dimensional vector $\mathbf{c}$ with c_0 being their gold holding and c_1 being their corn holding. Given the exchange ratio above, we can assume that (1,0) and (0,100) are points of equal worth and assuming that commodity space is a vector space thus

$$\delta((0,0),(1,0)) = \delta((0,0),(0,100)) \tag{1.2}$$

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This obviously does not meet equation 1.1 but if we re-normalise the corn axis by dividing by its price in gold, we get a metric

$$\delta_c(\mathbf{p},\mathbf{q}) = \sqrt{(p_0 - q_0)^2 + (\frac{p_1 - q_1}{100})^2} \tag{1.3}$$

which meets the equation we want for our two extreme points:

$$\delta_c((0,0),(1,0)) = \delta_c((0,0),(0,1)) \tag{1.4}$$

If this is our metric, then we can define a set of commodity holdings that are the same distance from the origin as holding 1oz of gold. Let us term this U the unit circle in commodity space:

$$U = \{a \in U : \delta_c((0,0),a) = 1\} \tag{1.5}$$

Since these points are equidistant from the origin, where the agent holds nothing, they must be positions of equal worth, and that movements along this path must not alter the net worth of the agent. Let us consider a point on U, where the agent holds $\frac{1}{\sqrt{2}}$oz gold and $\frac{100}{\sqrt{2}}$ bushels of corn.

Would this in reality be a point of equal worth to holding 1 oz of gold?

No, since the agent could trade their $\frac{100}{\sqrt{2}}$ bushels of corn for a further $\frac{1}{\sqrt{2}}$oz gold and end up with $\sqrt{2}$oz of gold. Thus there exists a point on U that is not equidistant from the origin, hence equation 1.3 can not be the form of the metric of commodity space and thus conjecture 1.1 falls, and commodity space is not a vector space.

2. THE METRIC OF COMMODITY SPACE

The metric actually observed in the space of bundles of commodities is:

$$\delta_b(\mathbf{p},\mathbf{q}) = \left|\sum \alpha_i (p_i - q_i)\right| \tag{2.1}$$

where $\mathbf{p}$, $\mathbf{q}$ are vectors of commodities, and α_i are relative values. The 'unit circle' in this space actually corresponds to a pair of parallel hyperplanes on above and one below the origin. One such hyperplane is the set of all commodity combinations of positive value 1 and the other, the set of all commodity combinations of value -1. The latter corresponds to agents with negative worth, i.e., net debtors.

Because of its metric, this space is not a vector space and it is questionable whether measures of similarity based on vector space metrics are appropriate for it. However it is possible to posit an underlying linear vector space of which commodity space is a representation.

3. Commodity Amplitude Space

We will now develop the concept of an underlying space, commodity amplitude space, which can model commodity exchanges and the formation of debt. Unlike commodity space itself, this space, is a true vector space whose evolution can be modeled by the application of linear operators. The relationship between commodity amplitude space and observed holdings of commodities by agents is analogous to that between amplitudes and observables in quantum theory.

Let us consider a system of n agents and m commodities, and represent the state of this system at an instance in time by a complex matrix $\mathbf{A}$, where a_{ij} represents the amplitude of agent i in commodity j. The actual value of the holding of commodity j by agent i , we denote by h_{ij} an element of the holding matrix $\mathbf{H}$. This is related to a_{ij} by the equation $a_{ij} = \sqrt{h_{ij}}$.

3.1. Commodity exchanges.

We can represent the process of commodity exchange by the application of rotation operators to $\mathbf{A}$. An agent can change the amplitudes of their holdings of different commodities by a rotation in amplitude space. Thus an initial amplitude of 1 in gold space by an agent can be transformed into an amplitude of 1 in corn space by a rotation of $\frac{\pi}{2}$. Borrowing Dirac notation we can write these as 1|gold>, and 1|corn>. A rotation of $\frac{\pi}{4}$ on the other hand would move an agent from a pure state 1|gold> to a superposition of states $\frac{1}{\sqrt{2}}|\text{gold}> + \frac{1}{\sqrt{2}}|\text{corn}>$. Unlike rotation operators in commodity space this is value conserving since on squaring we find their assets are now £$\frac{1}{2}$gold + £$\frac{1}{2}$corn.

The second conservation law that has to be maintained in exchange is conservation of the value of each individual commodity, there must be no more or less of any commodity after the exchange than there was before. This can be modeled by constraining the evolution operators on commodity amplitude space to be such that they simultaneously perform a rotation on rows and columns of the matrix $\mathbf{A}$.

Suppose we start in state:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \mathbf{H} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

Where agent zero has £1 of gold and no corn, and agent one has no gold and £4 of corn. We can model the purchase of £1 of corn by agent zero from agent one by the evolution of $\mathbf{A}$ to:

$$\mathbf{A2} = \begin{pmatrix} 0 & 1 \\ 1 & \sqrt{3} \end{pmatrix}$$

which corresponds to final holdings of:

$$\mathbf{H2} = \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$$

Note that the operation on amplitude space is a length preserving rotation on both the rows and the columns. The lengths of the row zero and column zero in **A2** are 1 the lengths of row and column one is 2 just as it was for **A.** This operation can be effected by the application of an appropriate rotation matrix so that $\mathbf{A2} = \mathbf{M}.\mathbf{A}$. A matrix which produces this particular set of rotations is:

$$\mathbf{M} = \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & \frac{\sqrt{3}}{2} \end{pmatrix}$$

3.2. **Price changes.** Price movements are equivalent to the application of scaling operations which can be modeled by the application of diagonal matrices. Thus a 50% fall in the price of corn in our model would be represented by the application of the matrix $\begin{matrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{matrix}$ to the current commodity amplitude matrix. Scaling operations are not length preserving.

3.3. **Modeling Debt.** We specified in section 3 that the amplitude matrix must be complex valued. This is required to model debt. Suppose that starting from holdings **H** agent zero buys £2 of corn from agent one. Since agent zero only has £1 of gold to pay for it, the transaction leaves the following holdings:

Agent	gold	corn
0	£-1	£2
1	£2	£2

The corresponding amplitude matrix is

$$\mathbf{A3} = \begin{pmatrix} i & \sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix}$$

It it interesting that this too is the result of applying a unitary rotation operator to the original amplitude vector since the length of row zero $|\mathbf{A3}_0| = i^2 + (\sqrt{2})^2 = 1$, likewise the lengths of all other rows and columns are preserved. The linear operator required to create debts has itself to be complex valued, thus if $\mathbf{A3} = \mathbf{NA}$ we have

$$\mathbf{N} = \begin{pmatrix} i & \frac{1}{\sqrt{2}} \\ \sqrt{2} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

4. Implications for similarity measures

Steedman [5] has proposed that a suitable criterion for assessing similarity of values to market prices is the angle between market price and value vectors, with small angles indicating closeness. If **m, v** are market price and value vectors respectively, the angle between them is given by:

$$ArcCos(\overrightarrow{\mathbf{m}}.\overrightarrow{\mathbf{v}})$$

where $\vec{\mathbf{v}}$ denotes the normalised value vector given by $\vec{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|}$.

If the argument in section 1 is accepted, we should consider using angles between price and value amplitude vectors instead. If we denote the normalised vectors in amplitude space by $\vec{\mathbf{m_a}}$and $\vec{\mathbf{v_a}}$, then the amplitude space angles are given by:

$$ArcCos(\overrightarrow{\mathbf{m_a}}.\overrightarrow{\mathbf{v_a}}^*)$$

where $\mathbf{x}^*$ is the conjugate of $\mathbf{x}$.

What will be the properties of this measure?

In general it will show smaller angles between vectors. For example suppose we have 3 commodities iron, corn, cotton as follows:

			amplitudes	
	value	price	value	price
corn	1	1	1	1
iron	3	2	$\sqrt{3}$	$\sqrt{2}$
cotton	1	2	1	$\sqrt{2}$
angle	30.2°		13.4°	

The fact that smaller angles are shown would be or little significance if the relative sizes of angles in the two spaces was the same. But this need not be the case. Consider the following example:

	value	price	PP	amplitudes		
				value	price	PP
corn	1	1	1	1	1	1
iron	$\frac{1}{2}$	-1	2	$\frac{1}{\sqrt{2}}$	i	$\sqrt{2}$
cotton	0.02	1	1	$\frac{\sqrt{2}}{10}$	1	1
θrelative to price in	74°	0	90°	54°	0	45°

Here we are comparing three hypothetical vectors of values, prices and prices of production (PP). If we treated commodity value space as a vector space, then prices of production would be orthogonal to market prices, wheras in amplitude space they are at 45° to market prices. When commodity space is treated as a vector space, values appear closer to market prices than do prices of production. When the assumption that commodity space is a vector space is dropped, then prices of production are closer to market prices.

5. Conclusion

We have argued that commodity space can not be directly modeled by a vector space, because of the metric it observes, but that it can be treated as the real valued representation or an underlying vector space. This complex vector space we have, following physics terminology, termed commodity amplitude space. Observed holdings of commodities and money by agents are the squares of corresponding commodity amplitudes. Commodity exchange relations, including the formation of commercial debt can be modeled by unitary rotation matrices operating on this amplitude space. The conceptual model presented borrows extensively from quantum formalism.

It is thus at least arguable that the empirical relation between market prices and labour values should be measured by the angles between their corresponding vectors in commodity amplitude space. The latter space, unlike commodity space, is a linear vector space within which angles of rotation have a clear meaning.

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