Popular Matchings

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and MATCH-UP 2015, University of Glasgow.

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- Input: a bipartite graph $G = (\mathcal{A} \cup \mathcal{B}, E)$.

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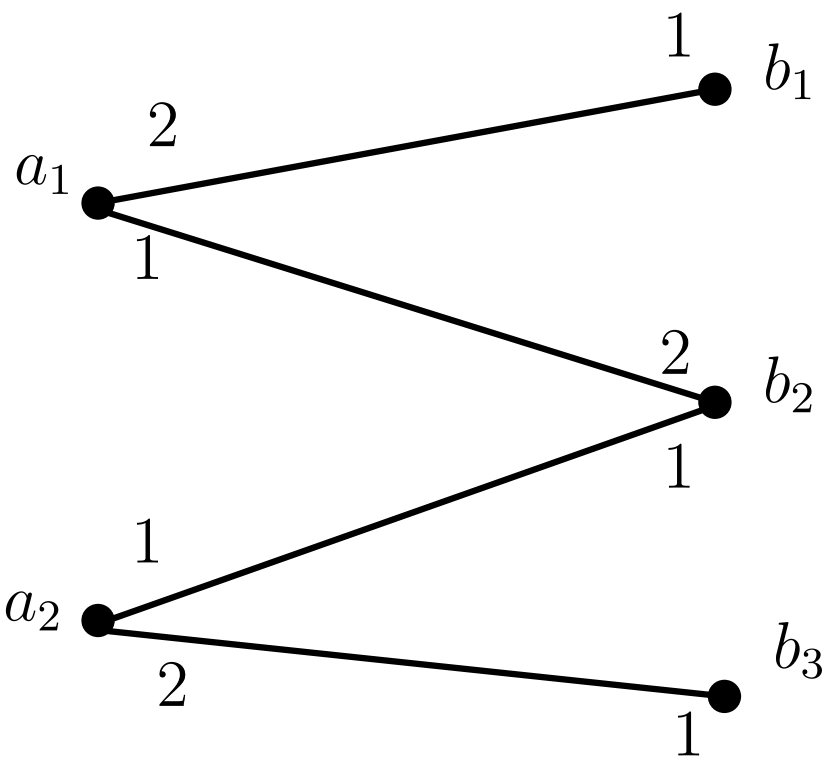

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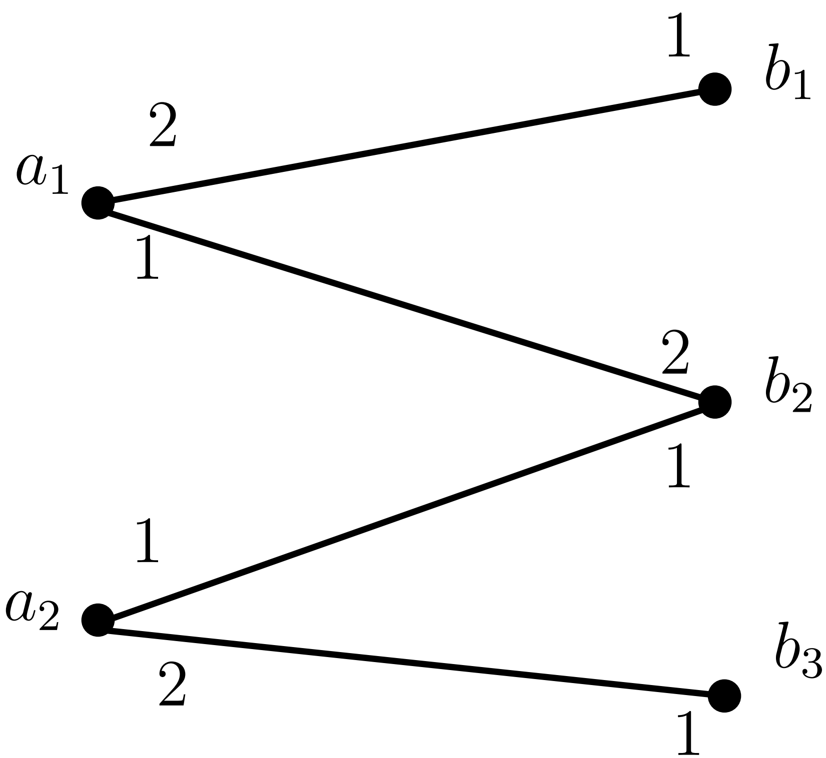

- $\mathcal{A}$: a set of students; $\mathcal{B}$: a set of advisers.

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    - M is a stable matching.

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  - all stable matchings in G have the same size.
  - |stable matching| could be as low as $|M_{max}|/2$.

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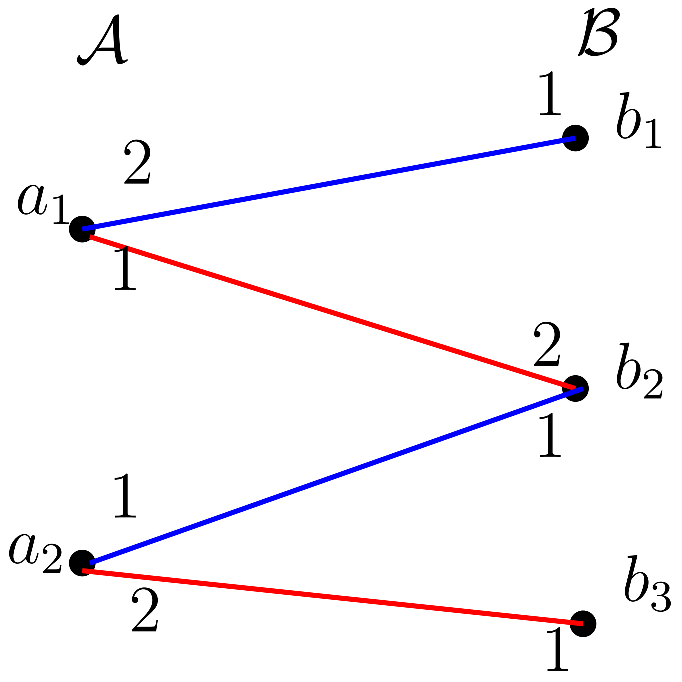
  - matching M_1 is *more popular* than matching M_2 if

$\#$ of vertices that prefer $M_1 > \#$ of vertices that prefer M_2.

An example



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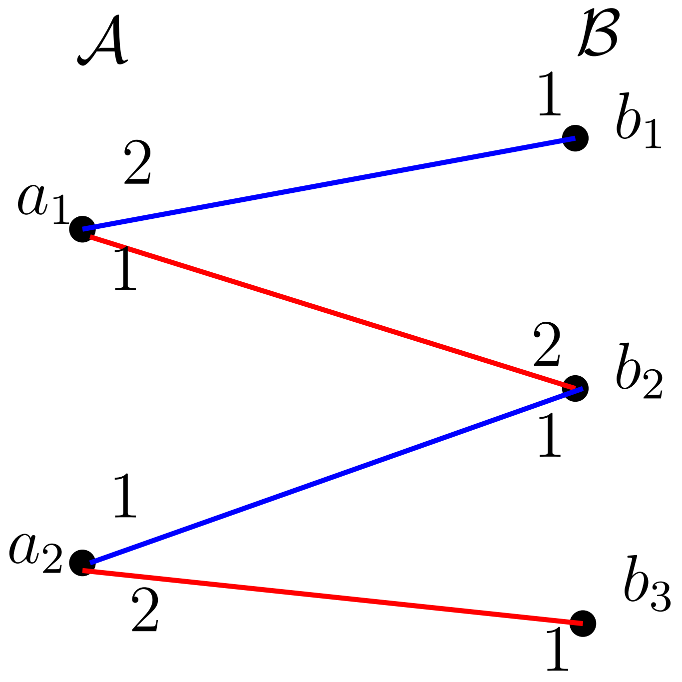

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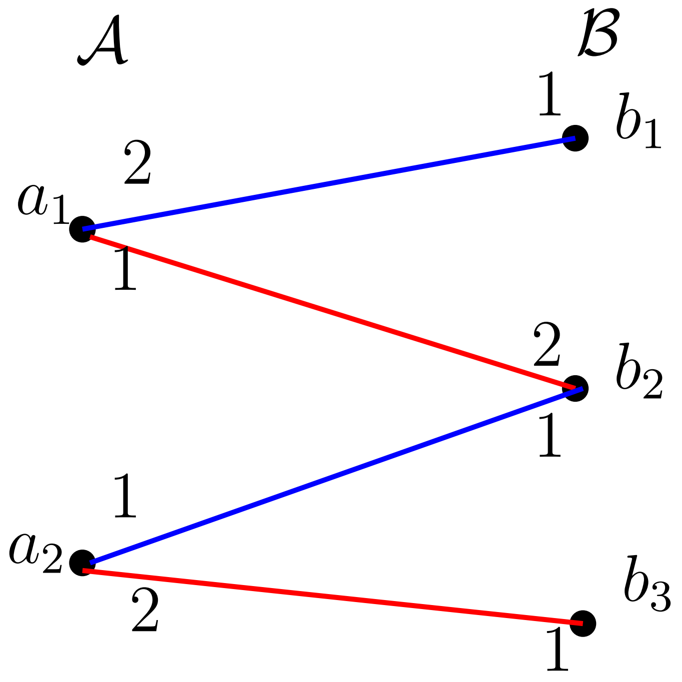

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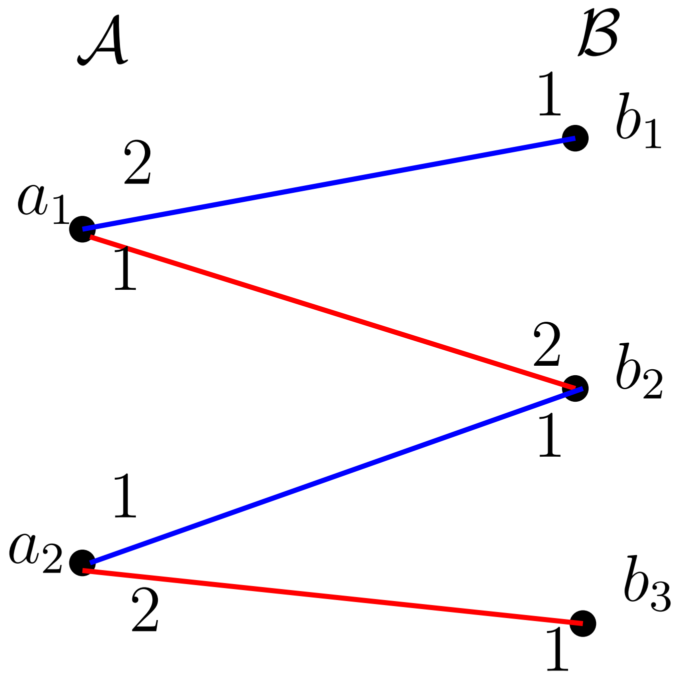

- a_1 and b_3 prefer the red matching
- b_1, b_2, and a_2 prefer the blue matching
- blue matching is more popular than red matching.

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M is popular $\Rightarrow$ for every matching M' we have:
$\#$ of vertices that prefer $M' \leq \#$ of vertices that prefer M.

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    - thus $\{$stable matchings$\} \subseteq \{$popular matchings$\}$.

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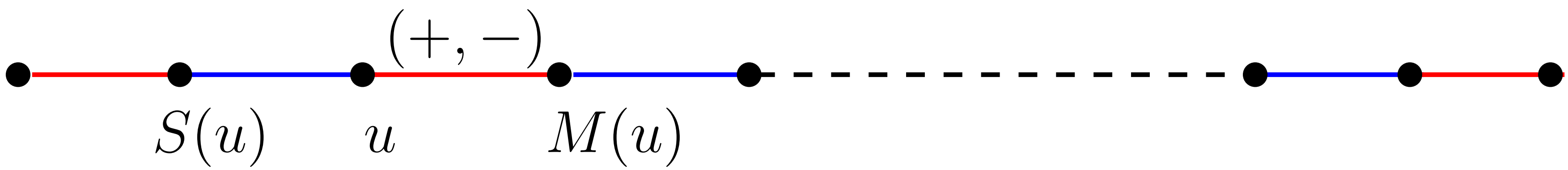

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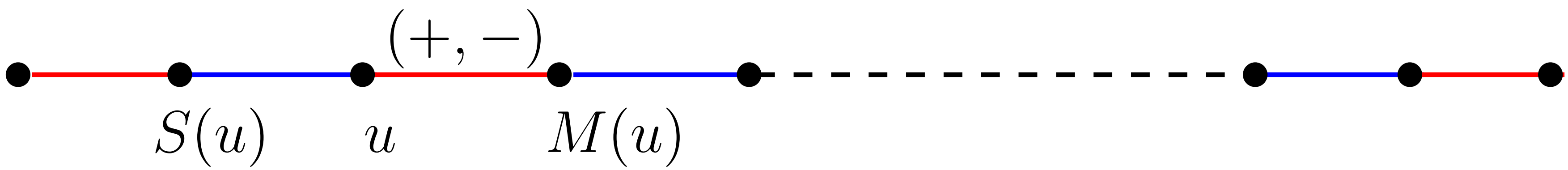

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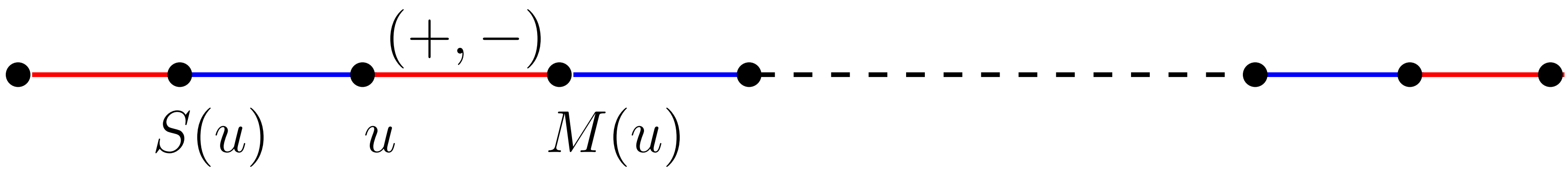

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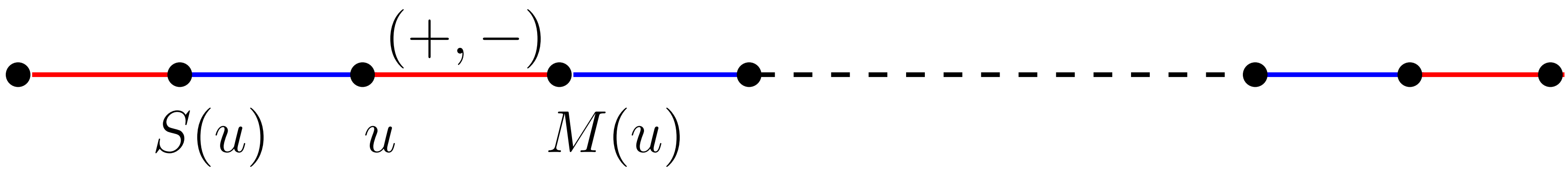

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    - so # of votes for $M \leq$ # of votes for S.

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- Let S be a stable matching and let M be a smaller matching.
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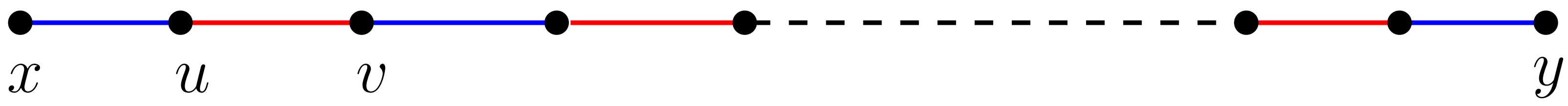
  - thus M is unpopular

The alternating path p



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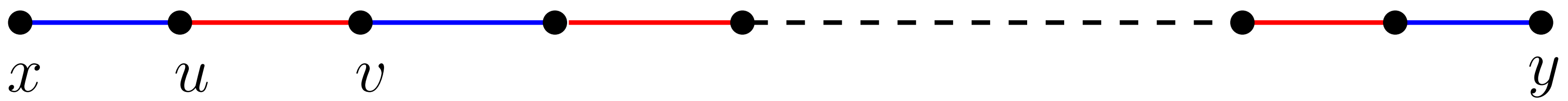

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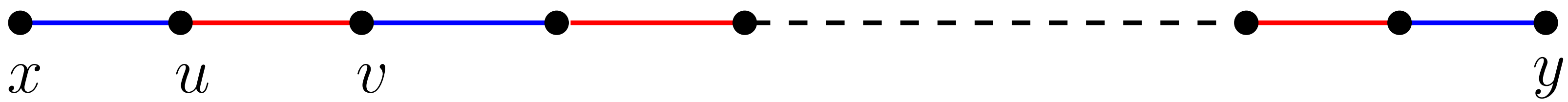

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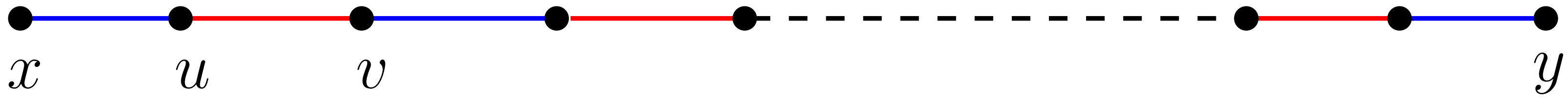

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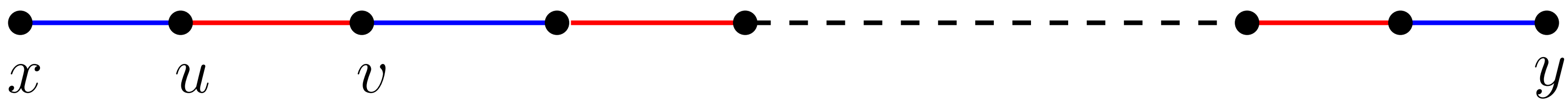

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- for every M-edge (u, v) in p:
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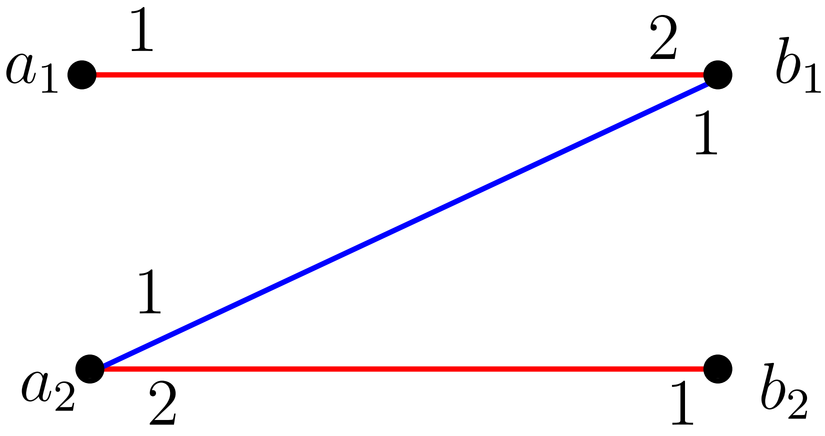
- Thus restricted to p, we have $S \succ M$.
  So $M \oplus p \succ M$.

Min vs max size popular matchings

- The blue matching is a minimum size popular matching.



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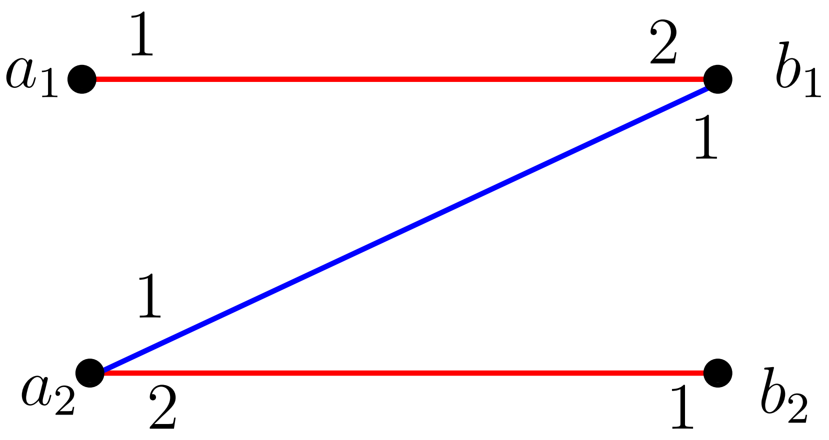

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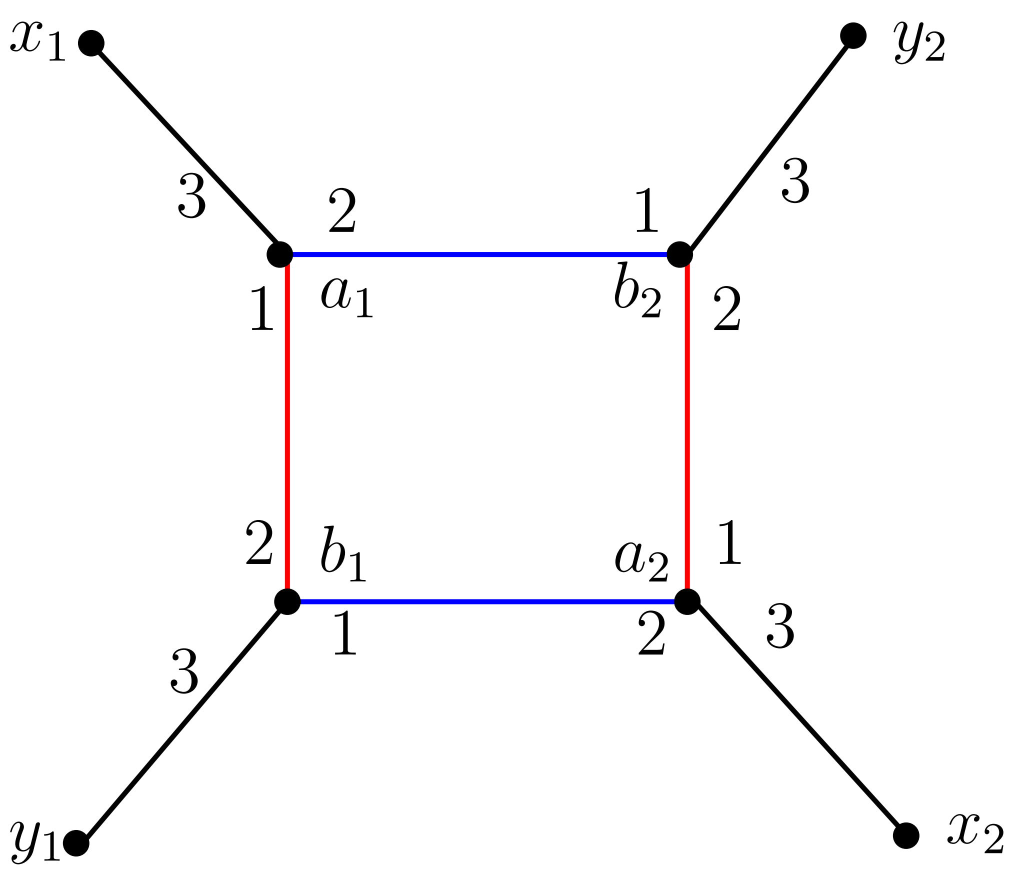
An interesting example

- Popular matchings of size 2 and size 4; none of size 3.



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- Delete from G all negative edges wrt M — call this graph G_M.

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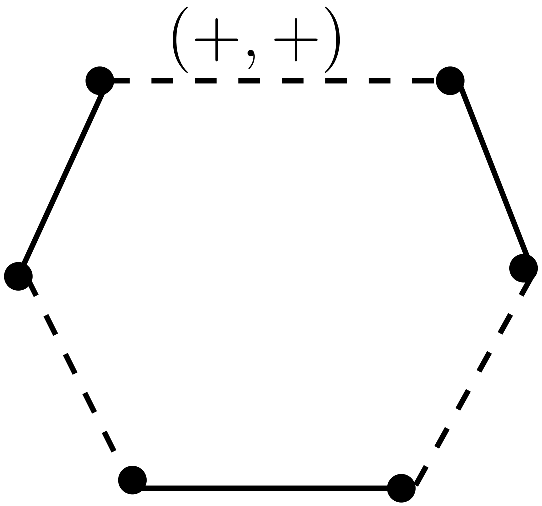

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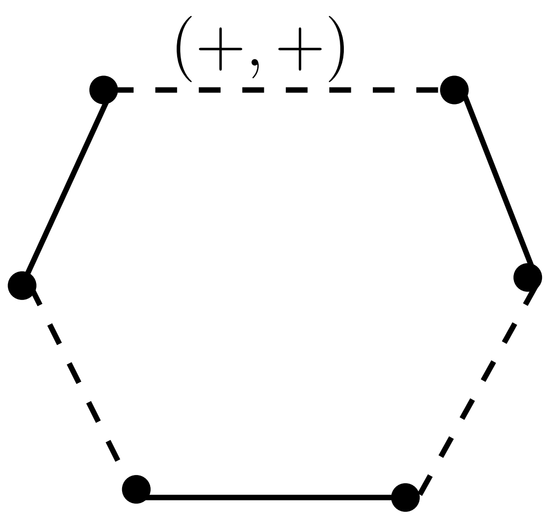

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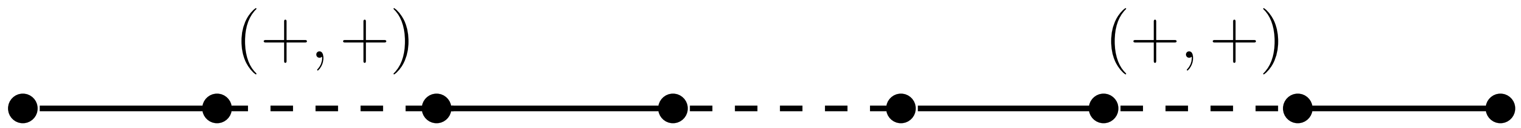

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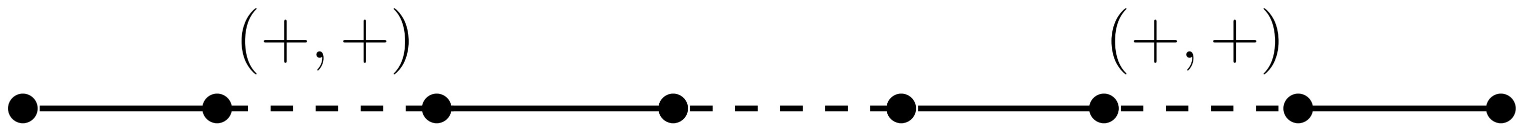

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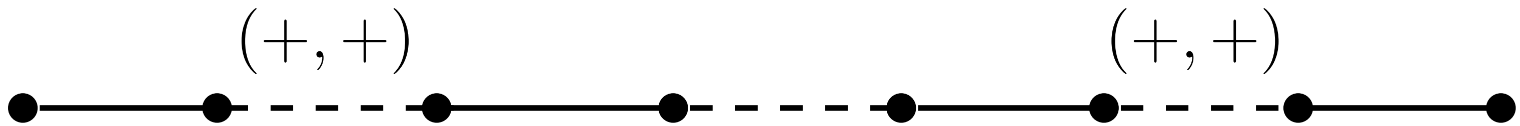

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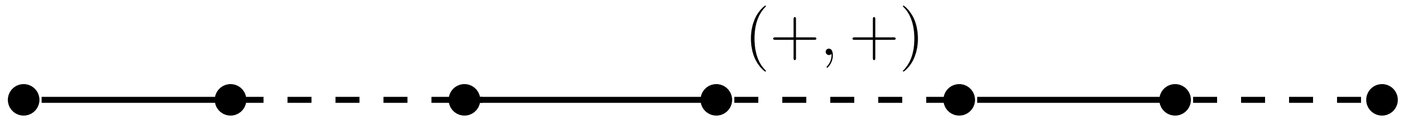

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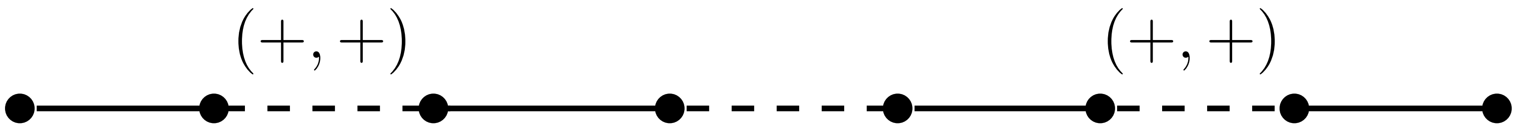

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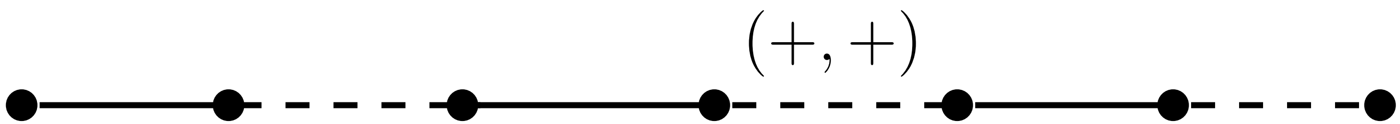

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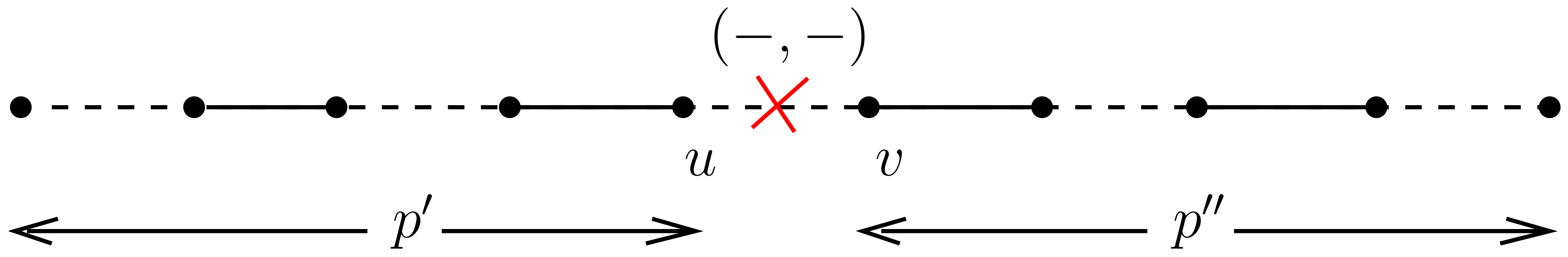

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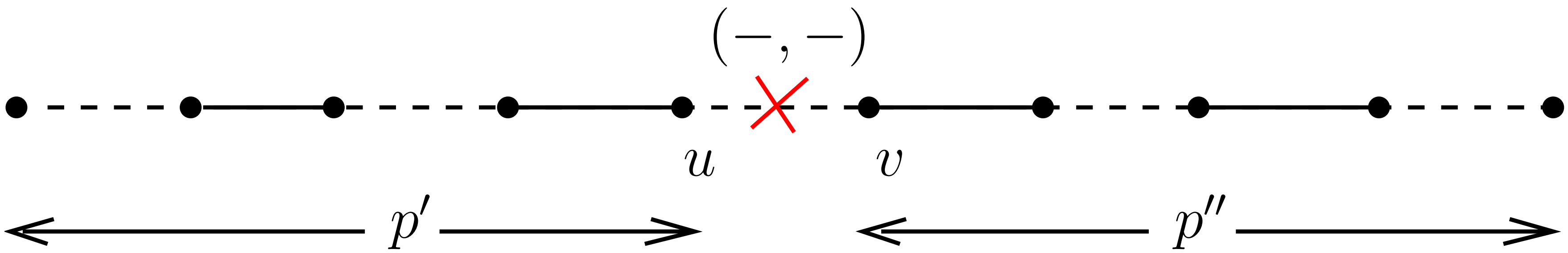

$\Rightarrow$ any larger matching M' has to be *unpopular*.

Max size popular matchings

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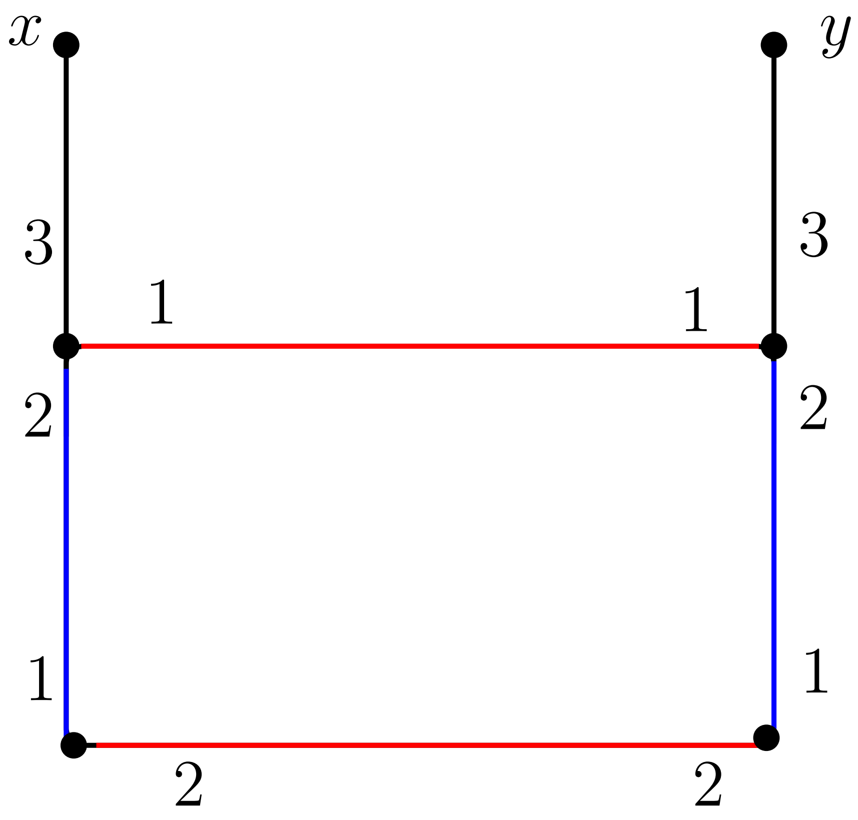

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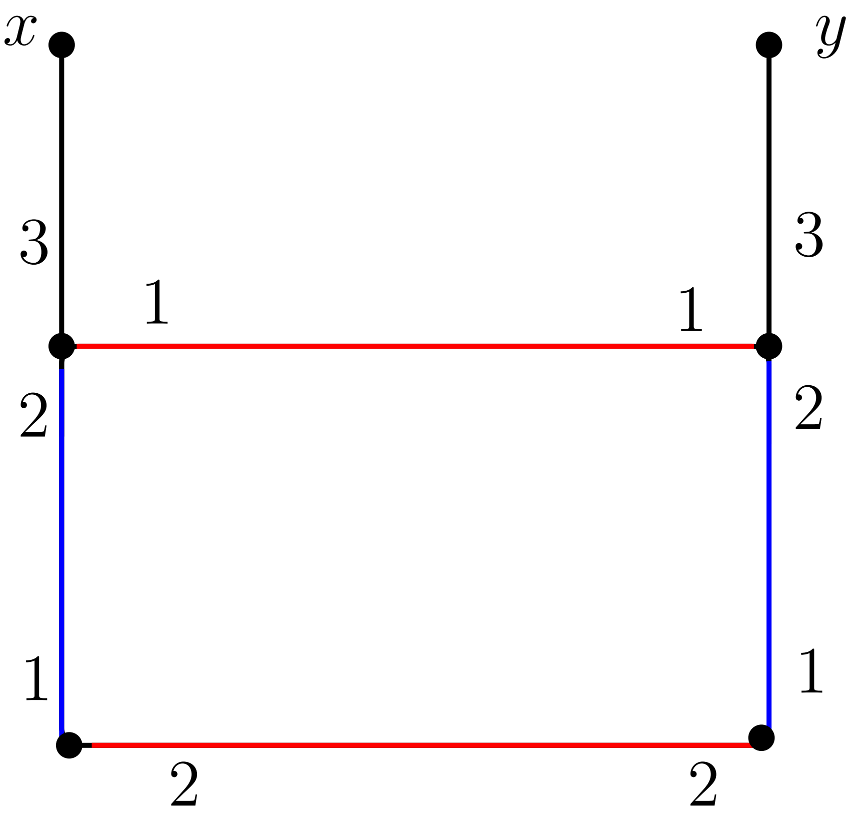

- G_M has an augmenting path wrt the red matching M.

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- An algorithm with running time $O(mn)$ to compute a max size popular matching in G. (Huang and K 2013)

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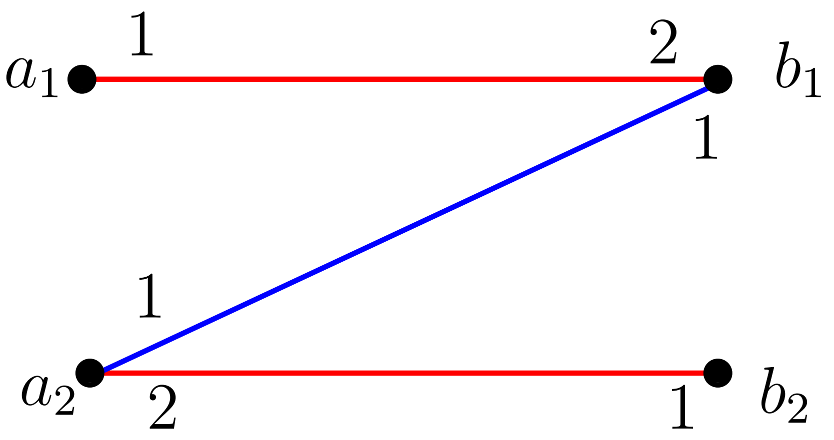

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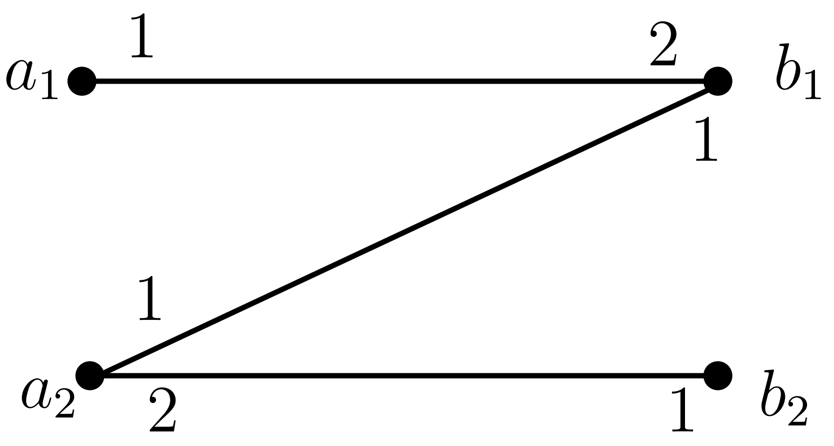

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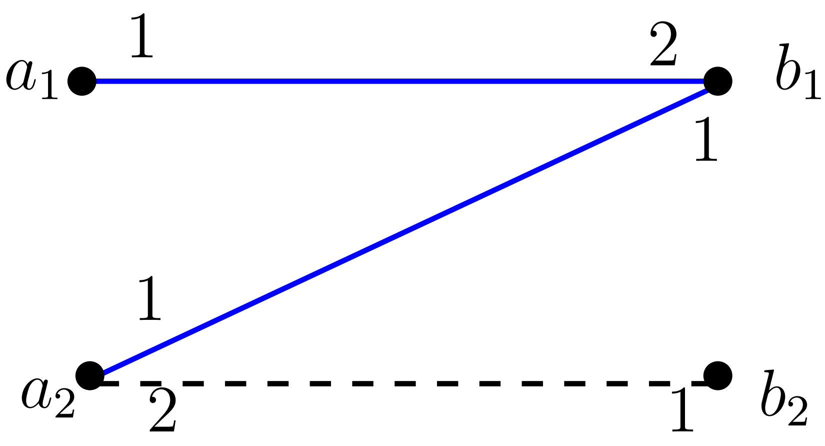
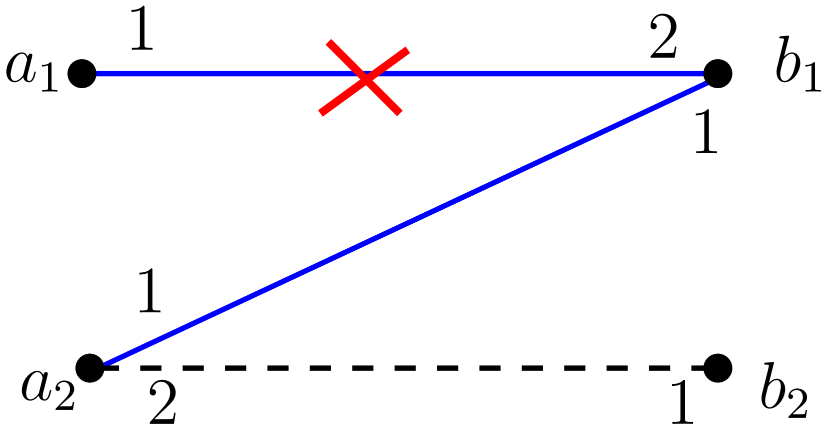


- a_1 proposes to his top neighbor b_1; so does a_2.

Gale-Shapley algorithm for stable matchings

- b_1 rejects a_1 and accepts a_2.



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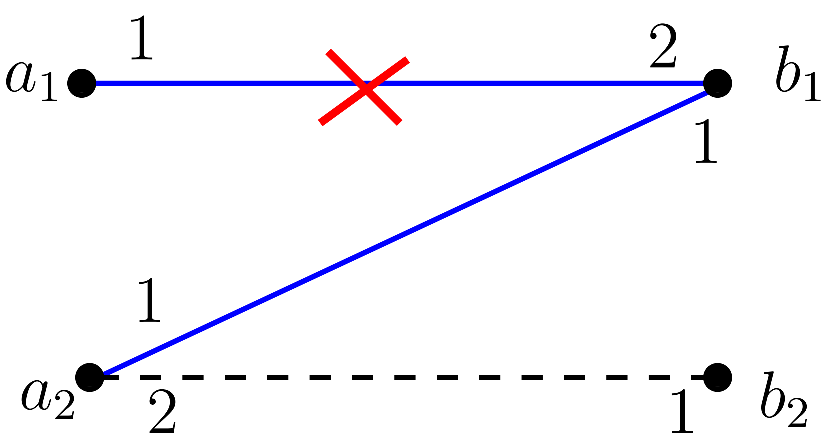
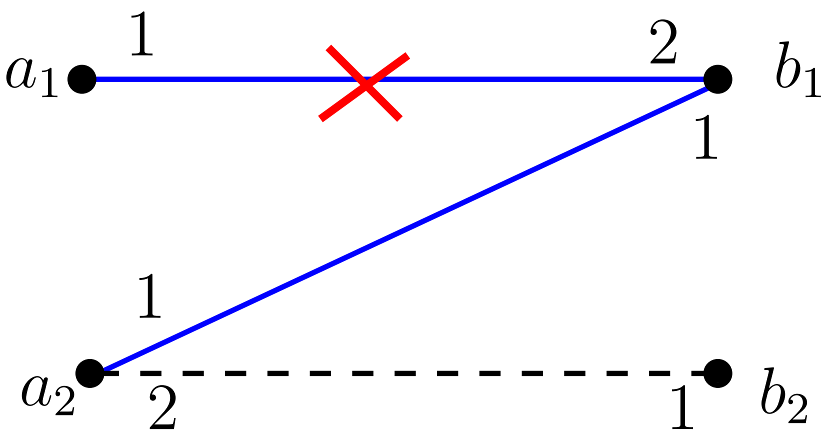


- The algorithm terminates when every man is either rejected by all his nbrs or gets matched to some nbr.

Modifying Gale-Shapley ...



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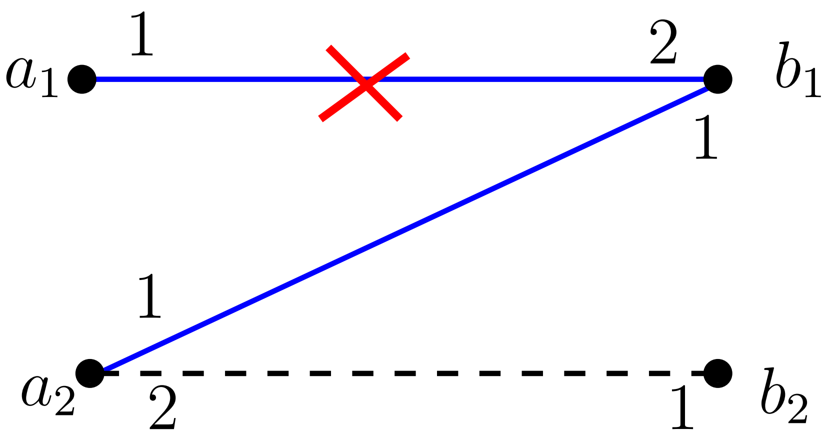

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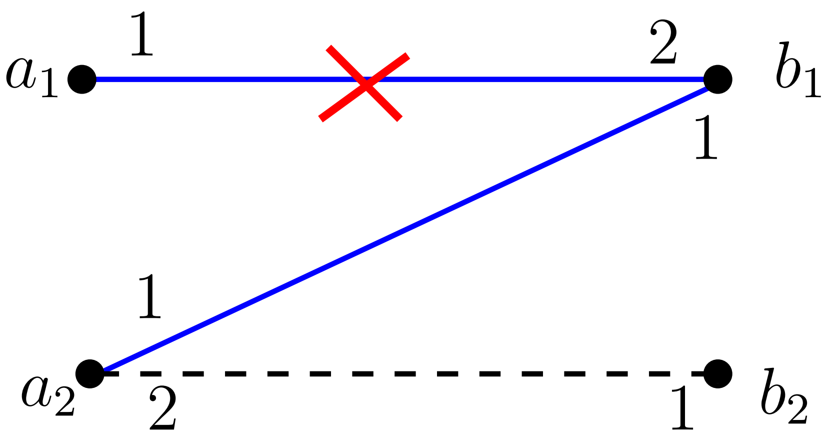

- Modify the Gale-Shapley algorithm so that a_1 gets a "second chance" to propose to b_1.
  - when a_1 proposes for the *second* time to b_1, then b_1 should prefer a_1 to a_2.

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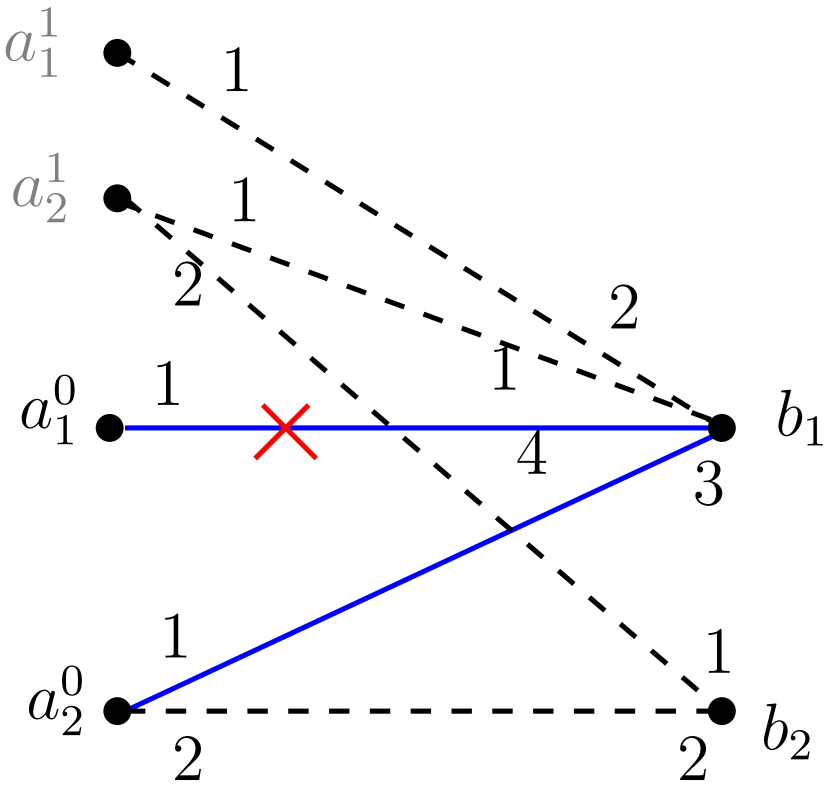

- a_1^0 is rejected by his only neighbor b_1.

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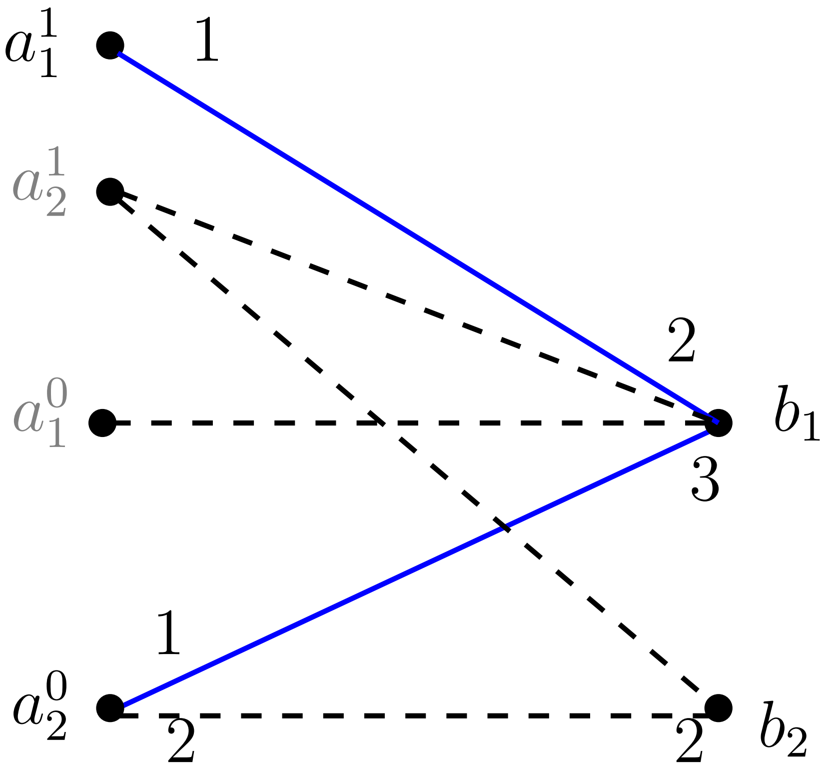

- So a_1^1 becomes active and proposes to b_1.

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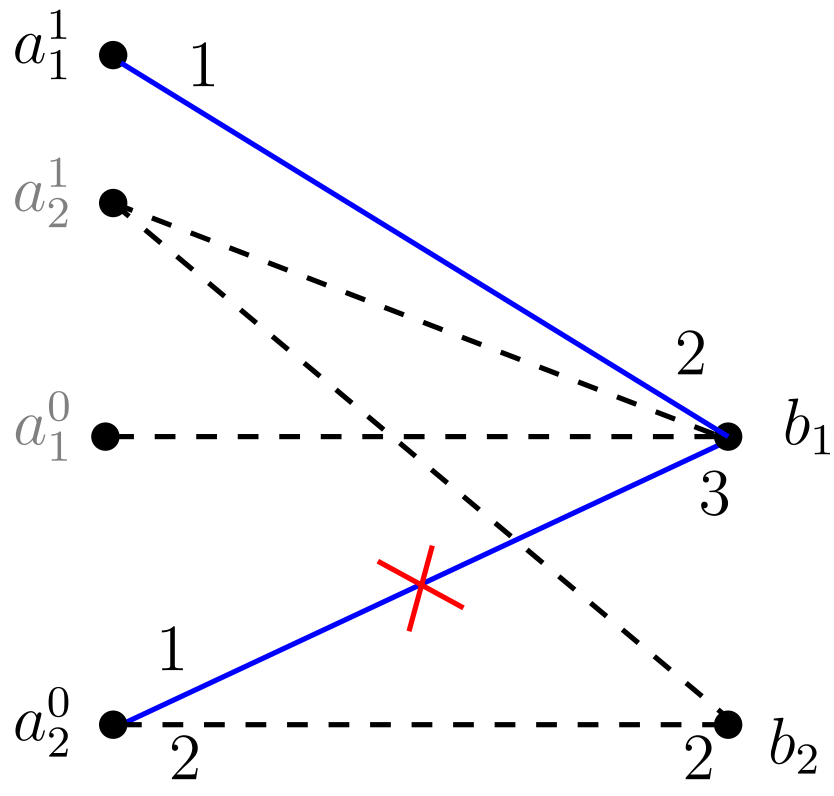

- b_1 accepts a_1^1 and rejects a_2^0.

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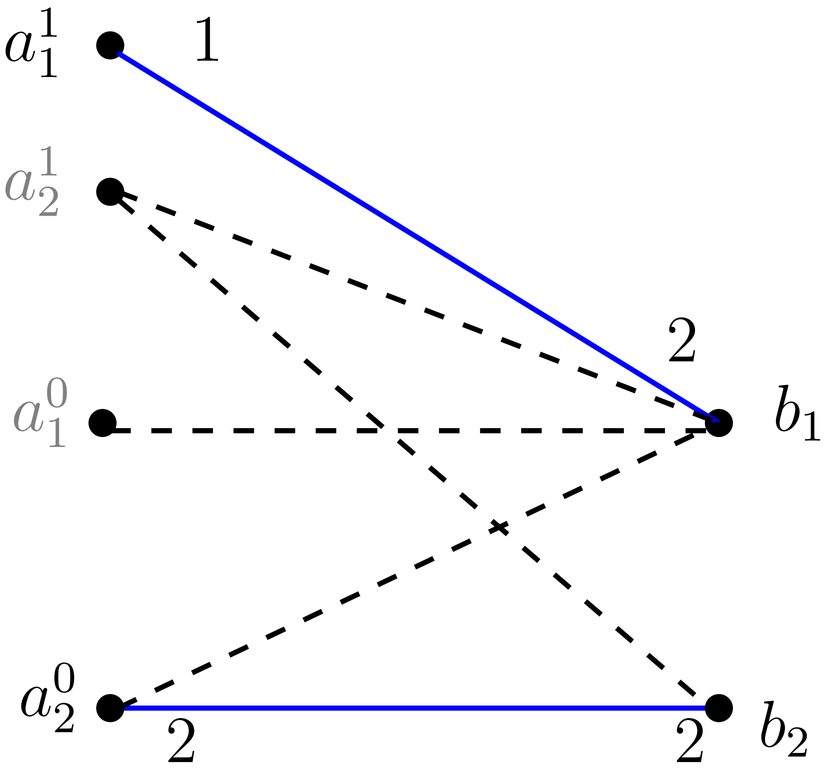
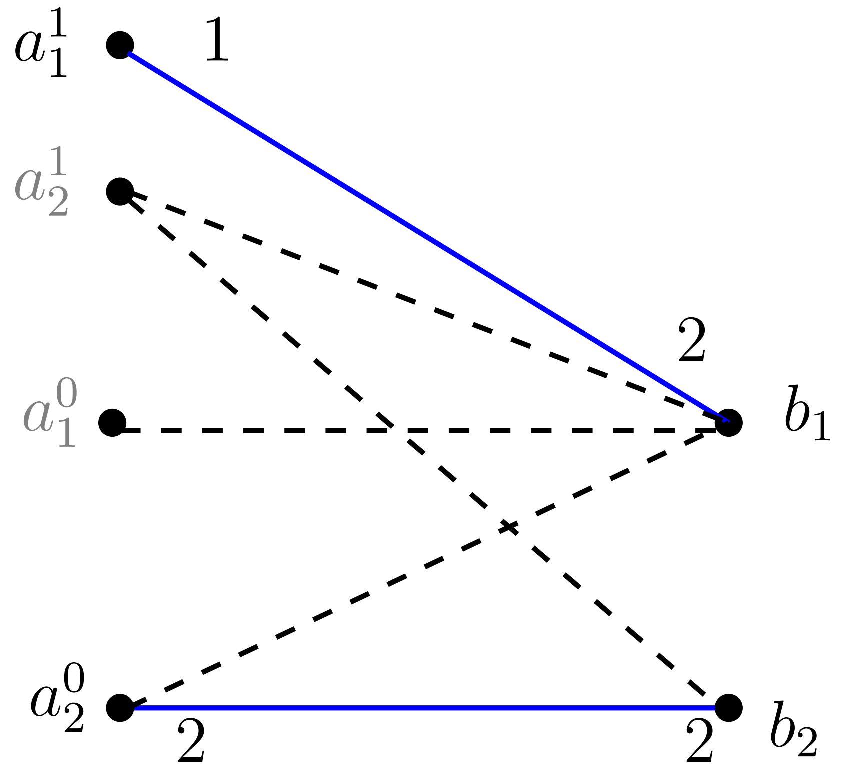


- So a_2^0 proposes to his next preferred neighbor b_2.

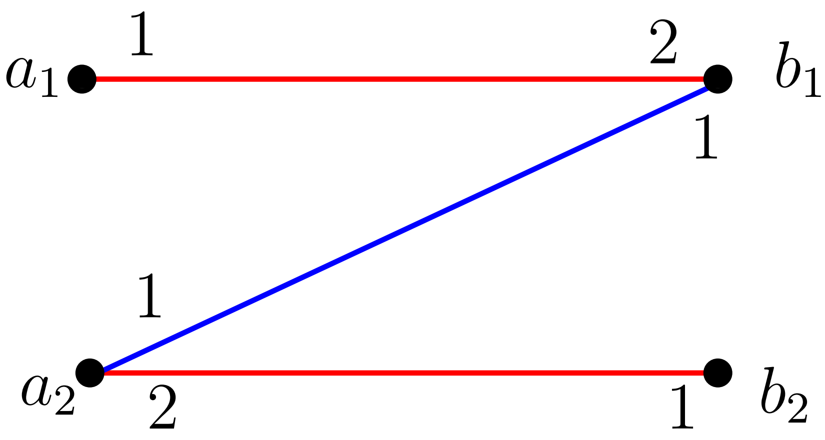
In the new graph



- The matching $\{(a_1^1, b_1),\ (a_2^0, b_2)\}$ is computed.

Back in the original graph

- Thus OPT $= \{(a_1, b_1), (a_2, b_2)\}$, the red matching, is found.



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    - introduce a_i^1 into the set of active vertices.

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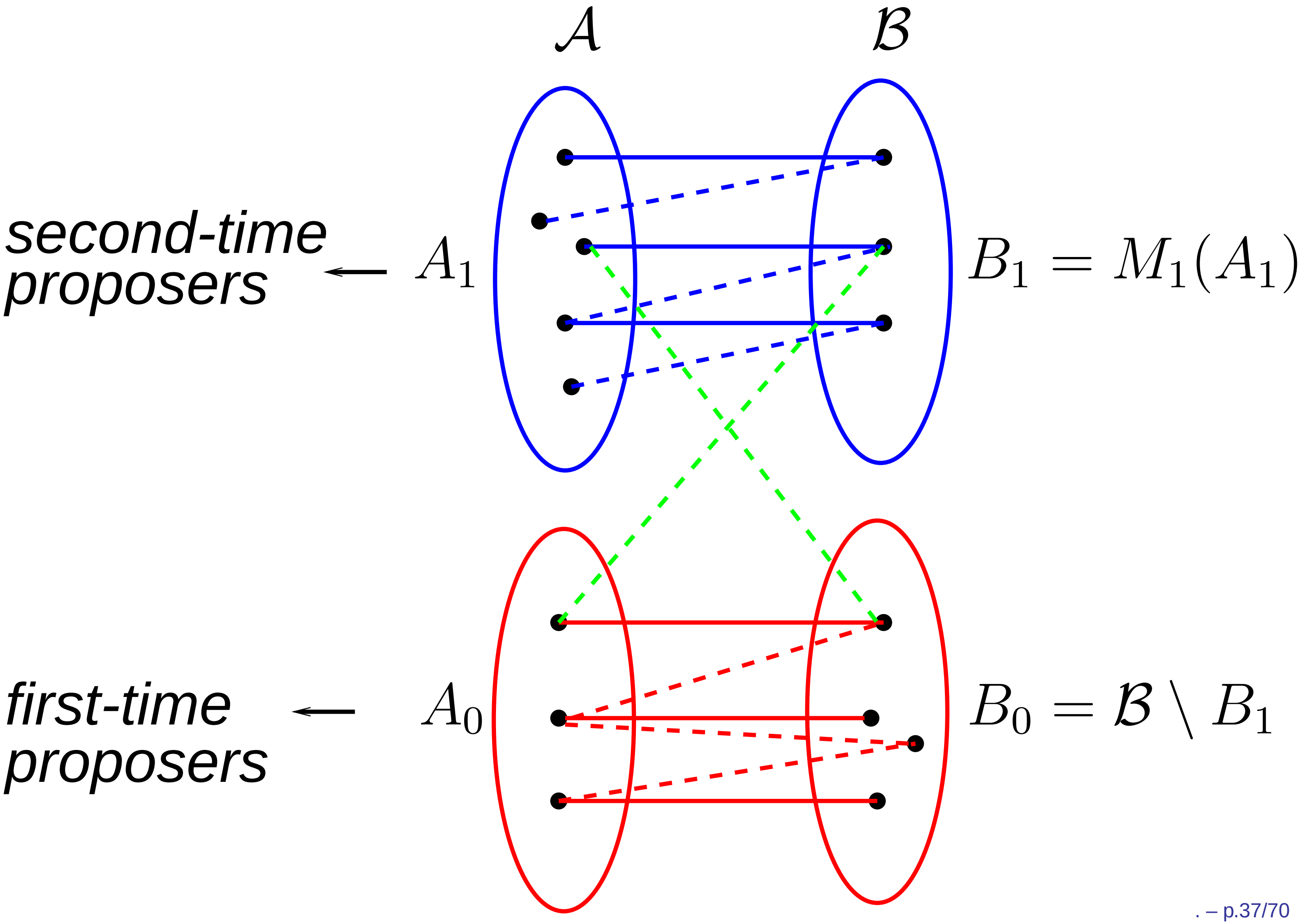
- Let M_1 be the matching computed by our algorithm.

Properties of M_1

- $M_1 \subseteq (A_0 \times B_0) \cup (A_1 \times B_1)$.



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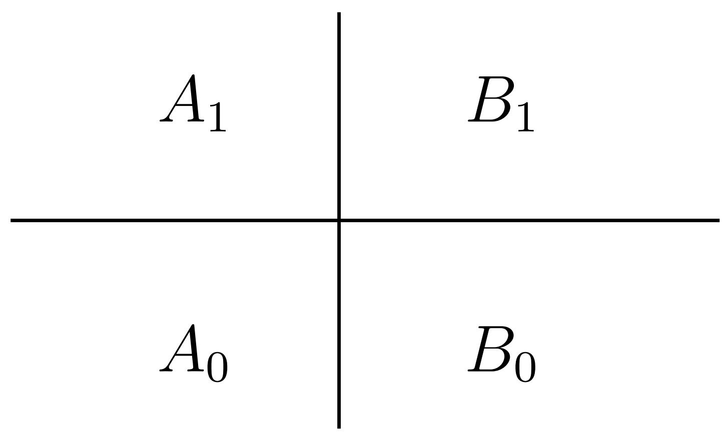

- Any blocking edge to M_1 has to be in $A_0 \times B_1$.

Partition of $\mathcal{A}$ and $\mathcal{B}$

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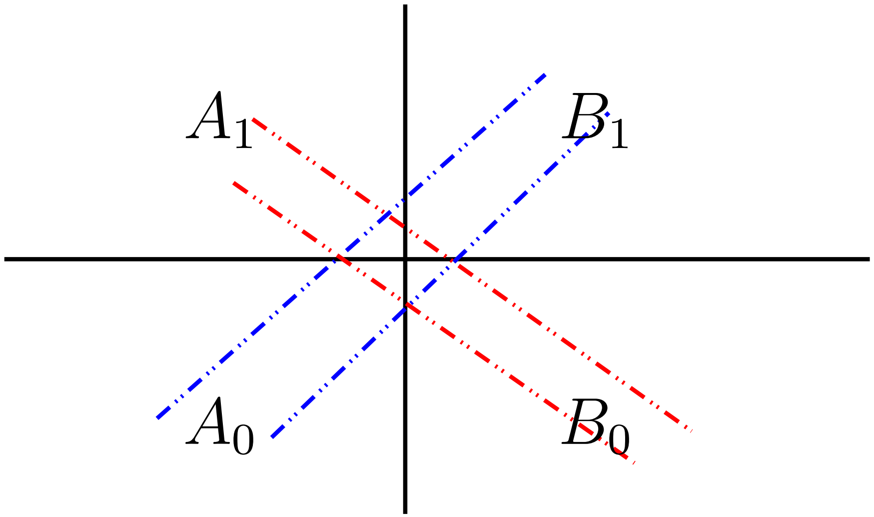

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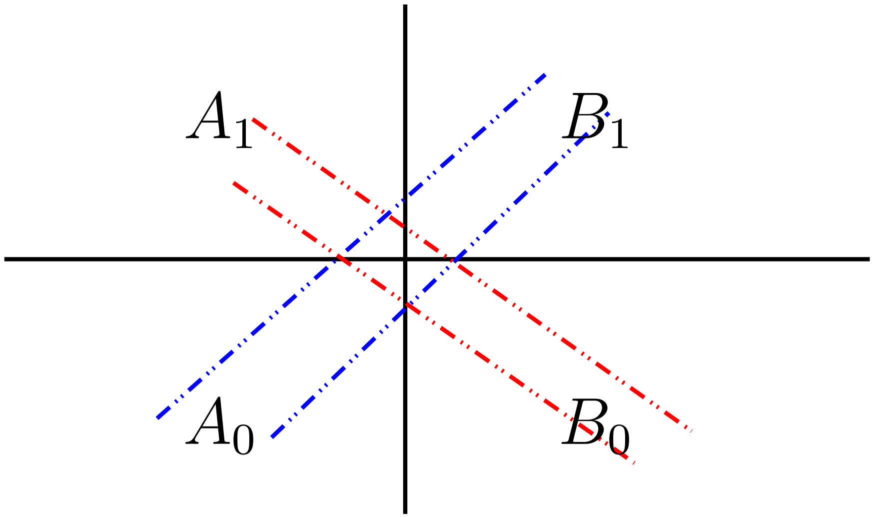

- Thus G_{M_1} has no edge in $A_1 \times B_0$.

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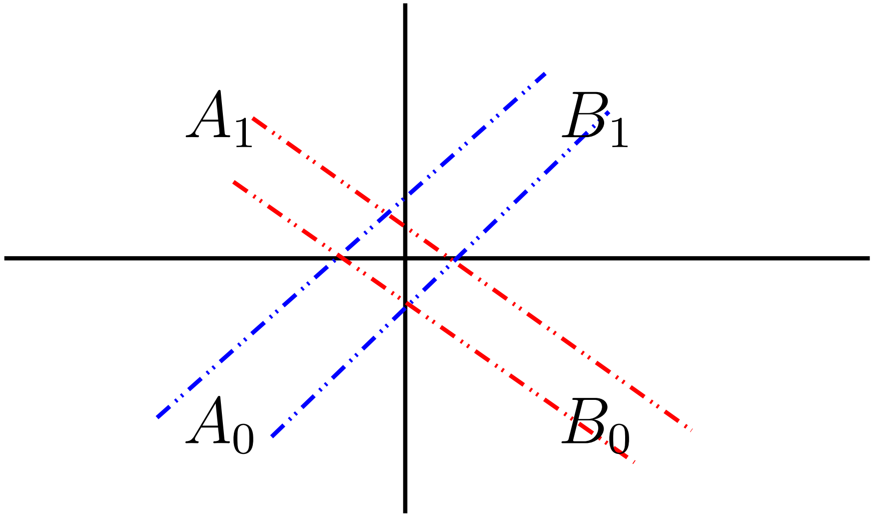

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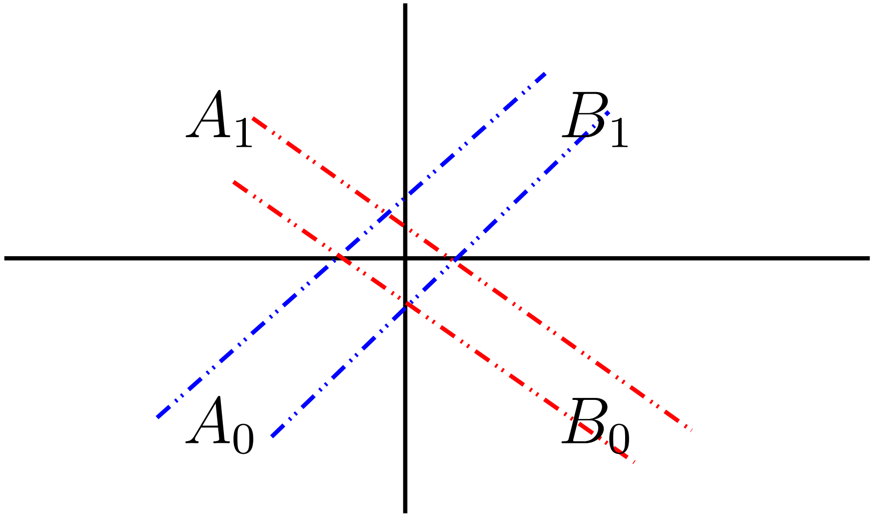

- There is *no* augmenting path wrt M_1 in G_{M_1}.

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- What about $|M_1|$ in terms of $|M_{max}|$?

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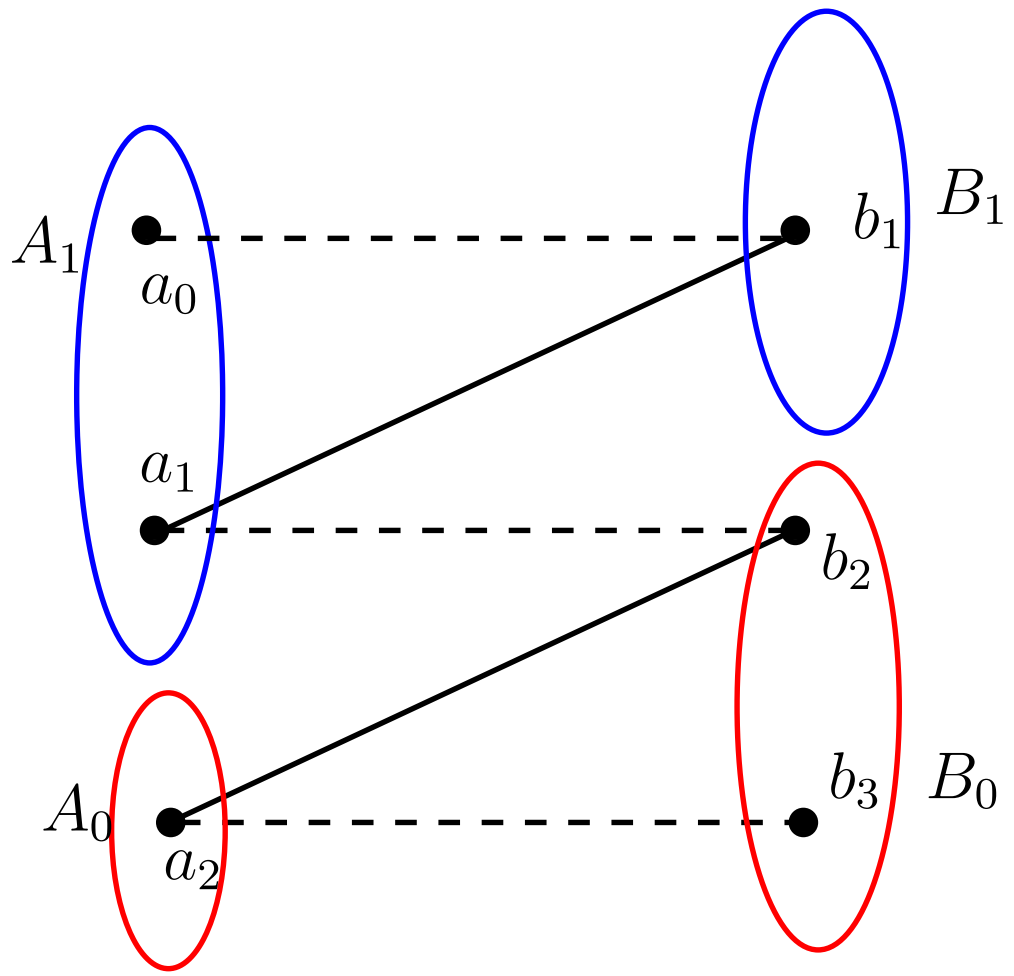

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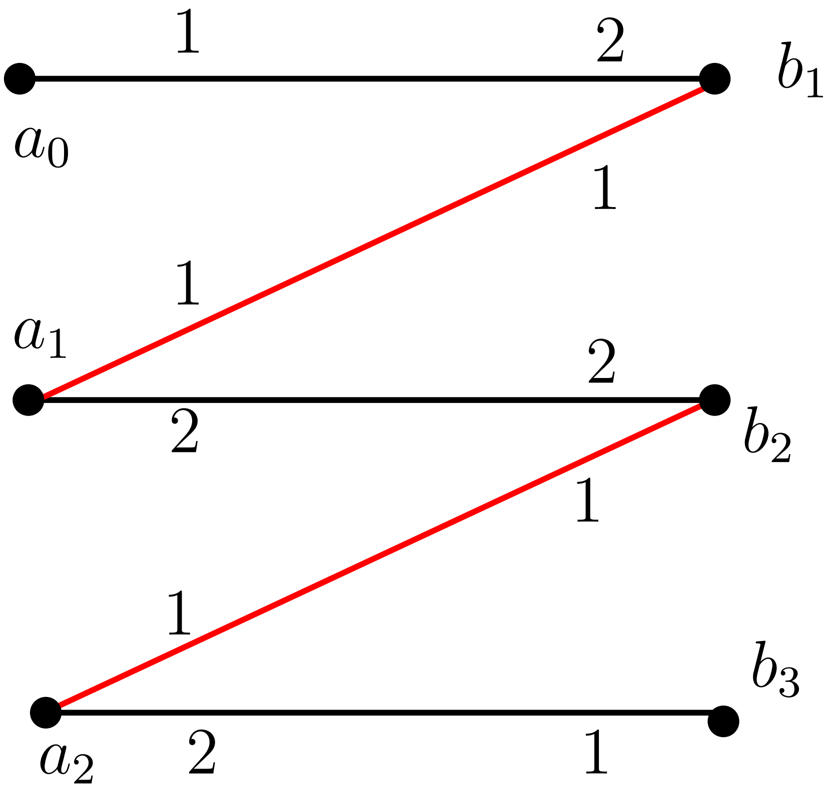
A tight example for the $2/3$ bound



- $|M_1| = 2$ while $|M_{max}| = 3$.

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$u(M) = \beta \Rightarrow$ for every matching M' we have:

|$\{$vertices that prefer $M'\} \leq \beta \cdot |\{$vertices that prefer $M\}|$.

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- Is there an $M^* \equiv$ a maximum cardinality matching s.t. for each maximum cardinality matching M: $M^* \succeq M$?

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    - for each $a \in \mathcal{A}$: at most one of $a^0, a^1, \ldots, a^{k-1}$ is active at any point.

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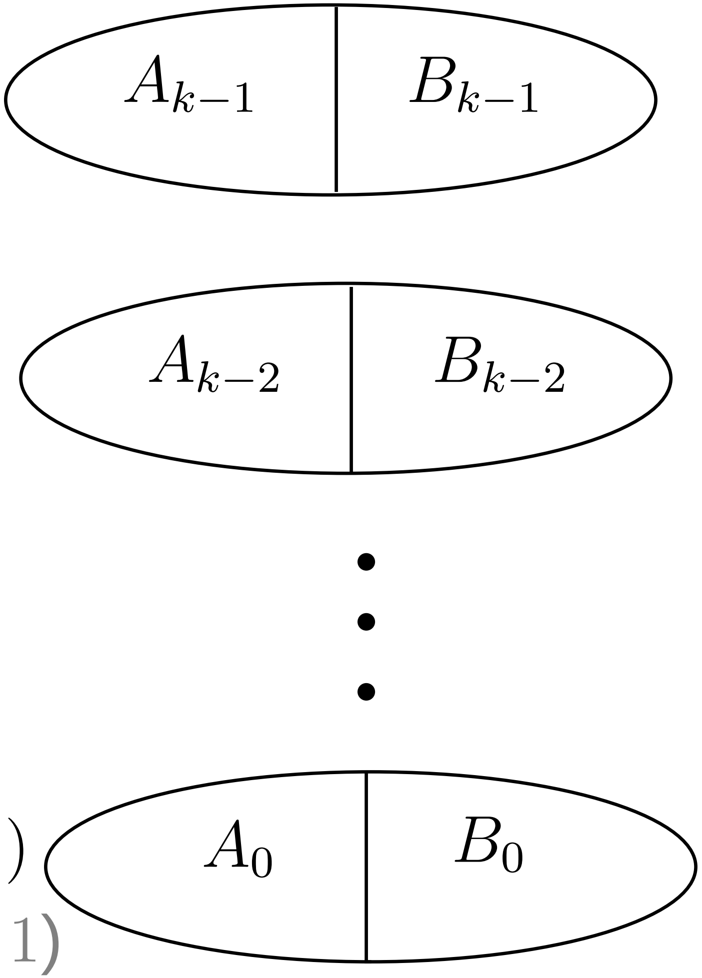
- Let M_{k-1} be the matching returned by this algorithm.

The partition of $\mathcal{A}$ and $\mathcal{B}$

- $A_i = \{a \in \mathcal{A}$ such that a is in level i at the end$\}$.

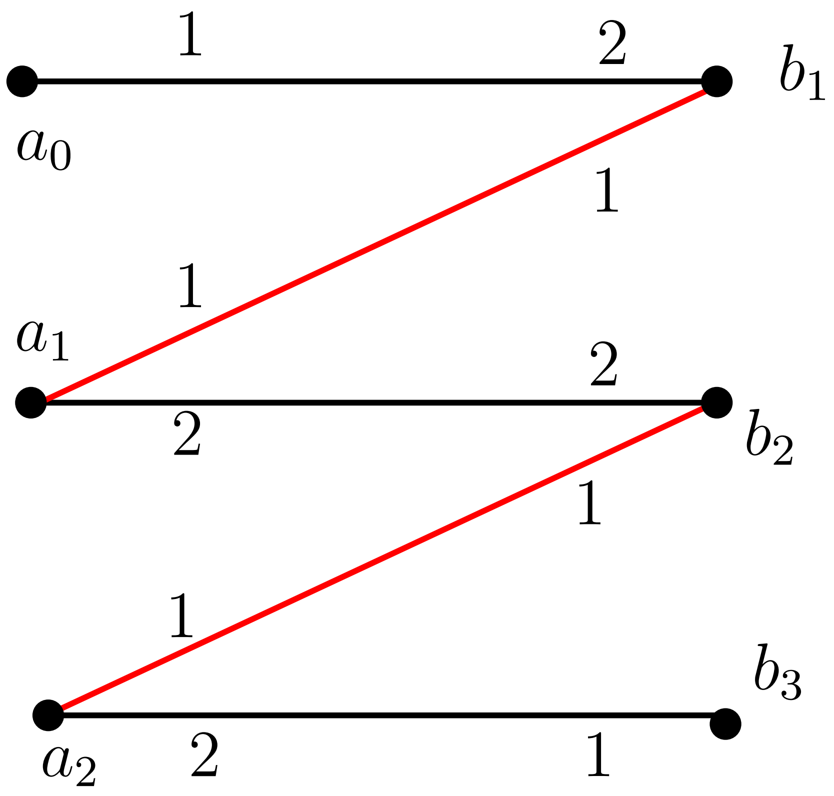


- $B_i = M_{k-1}(A_i)$

(for $1 \leq i \leq k-1$)

The 3-level algorithm

- Say we run the 3-level algorithm on our tight example for the 2-level algorithm ...



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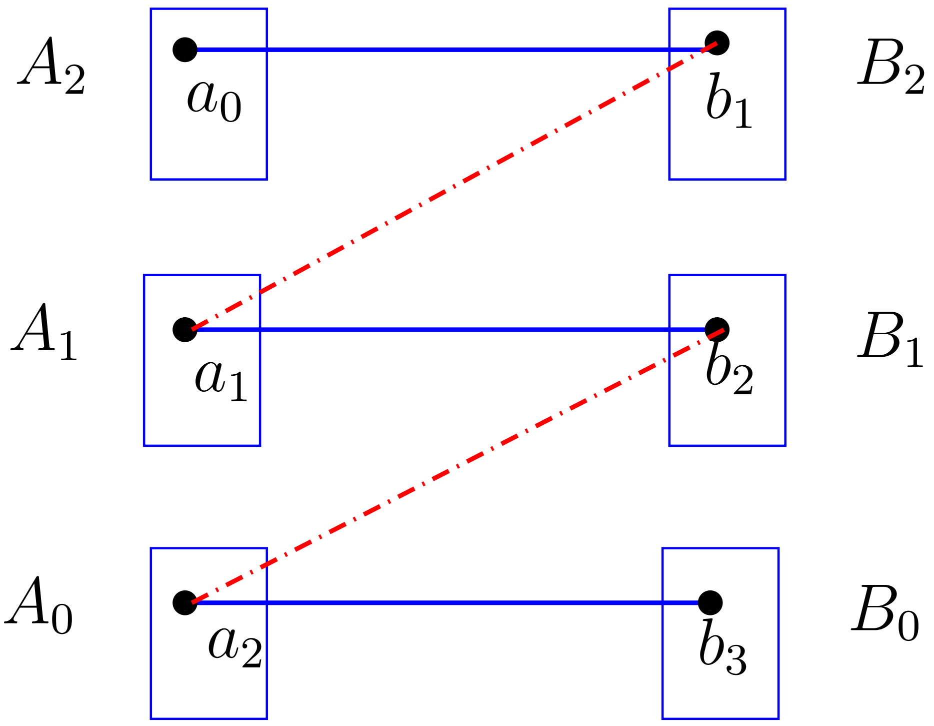

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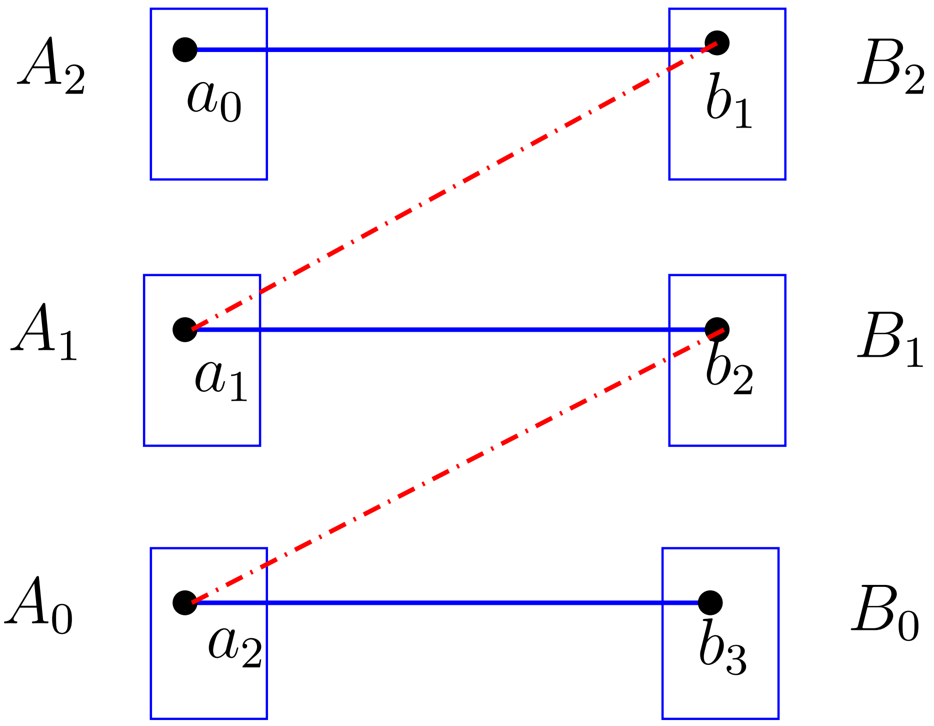
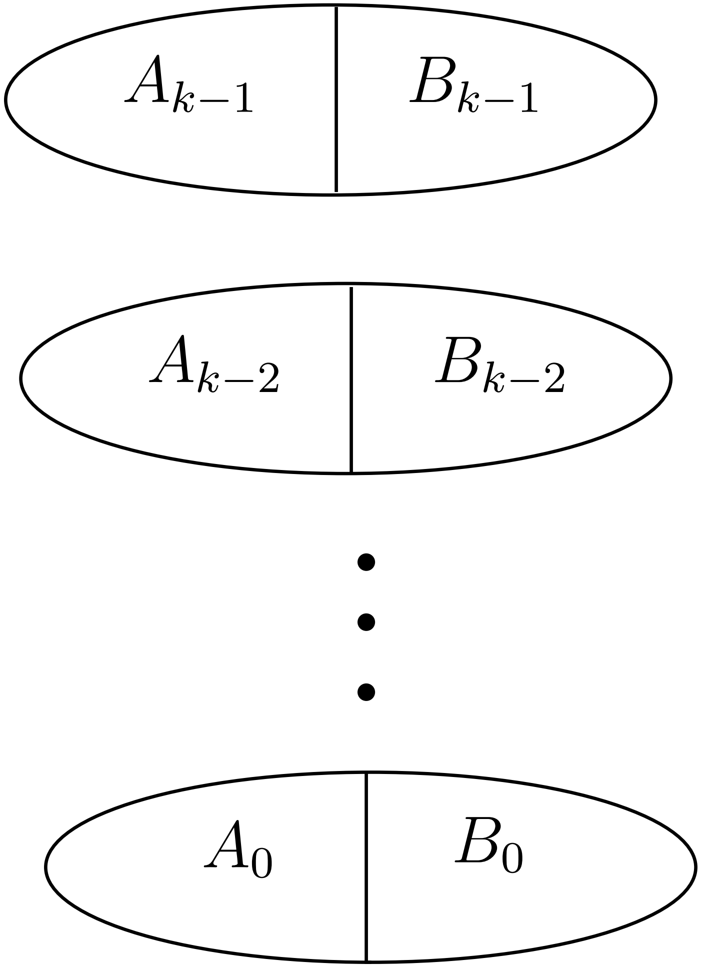


- The matching $M_2 = \{(a_0, b_1), (a_1, b_2), (a_2, b_3)\}$ is computed by the 3-level algorithm.

Properties of the matching M_{k-1}

- $M_{k-1} \subseteq (A_{k-1} \times B_{k-1}) \cup (A_{k-2} \times B_{k-2}) \cup \cdots \cup (A_0 \times B_0)$.



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    - in particular, $|M| > |M_{k-1}| \implies M_{k-1} \succ M$.

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- The problem is NP-hard even with one-sided ties. (Cseh, Huang, and K 2015)

One-sided ties

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    - the case where each $b \in \mathcal{B}$ puts all neighbors into a single tie has an $O(n^2)$ algorithm.

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$\mathcal{B} \setminus F$

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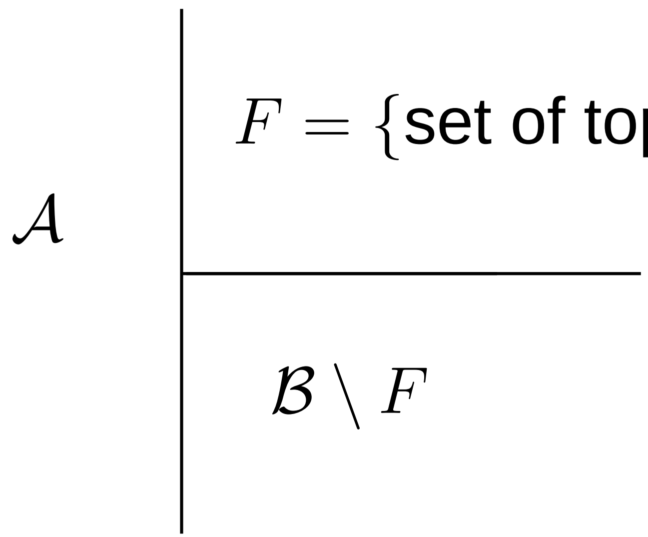

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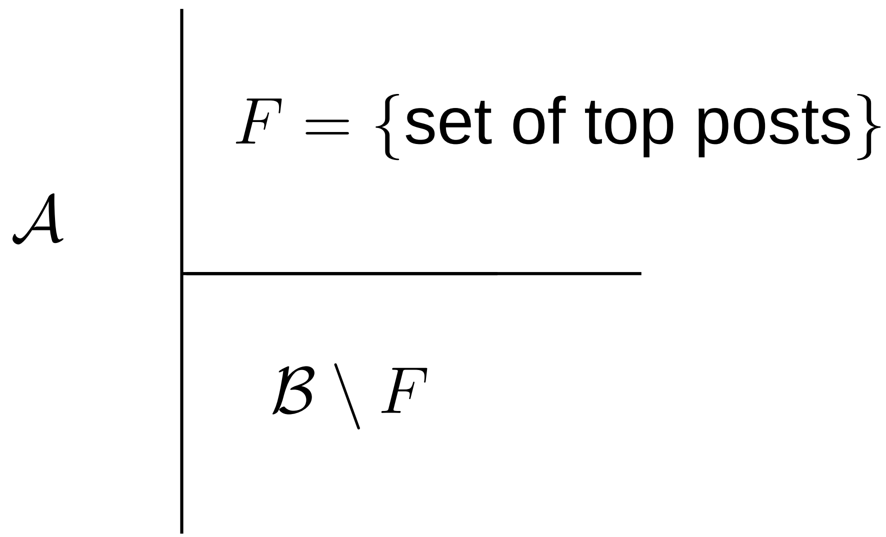

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- A linear time to solve the popular matching problem: extends to the case with ties in preference lists.

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    - $Z = \mathcal{B} \setminus (X \cup Y)$.

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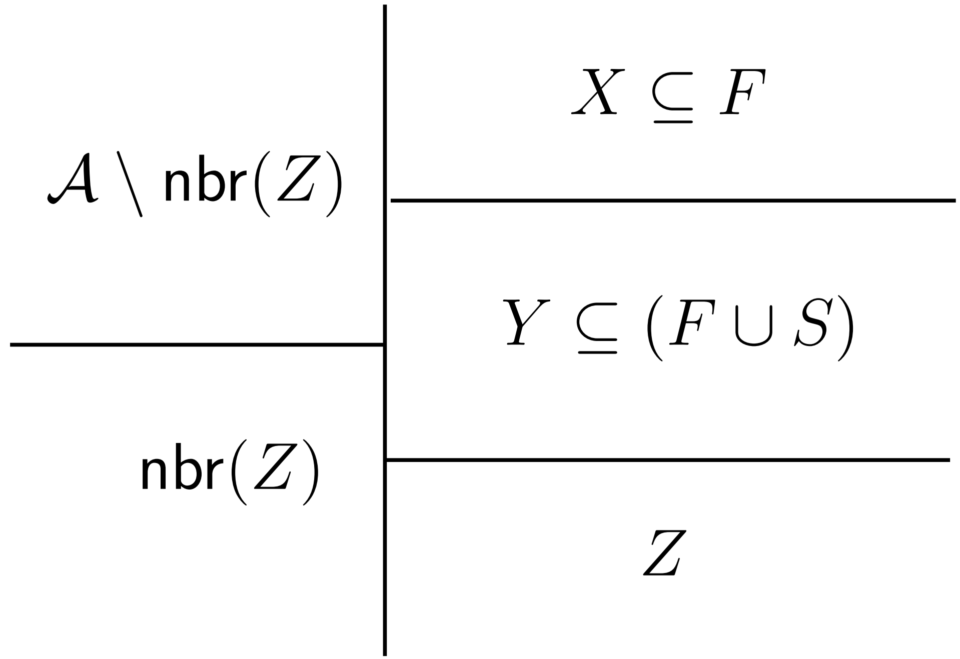

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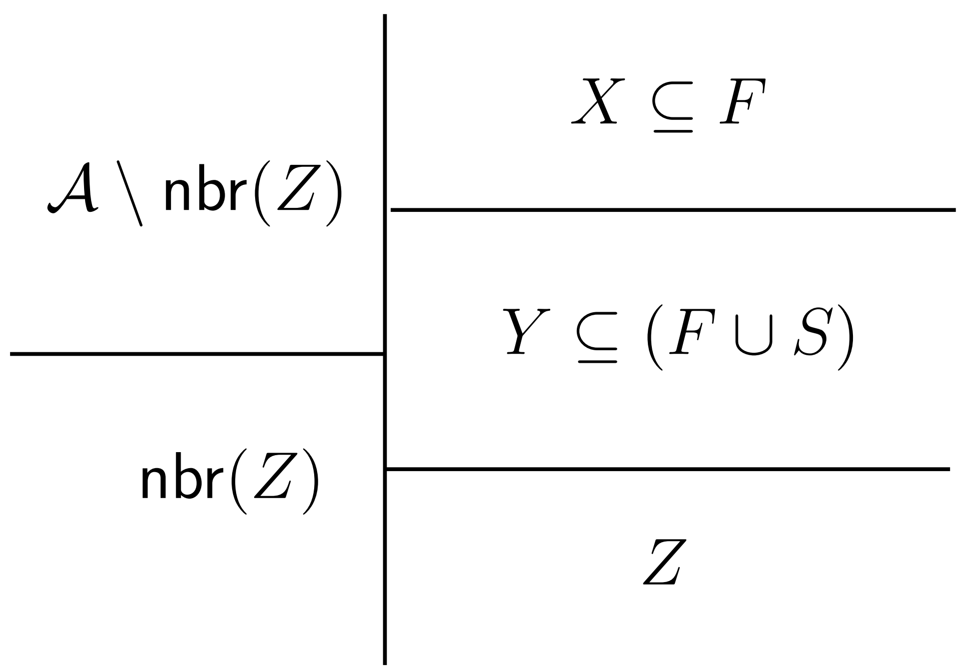

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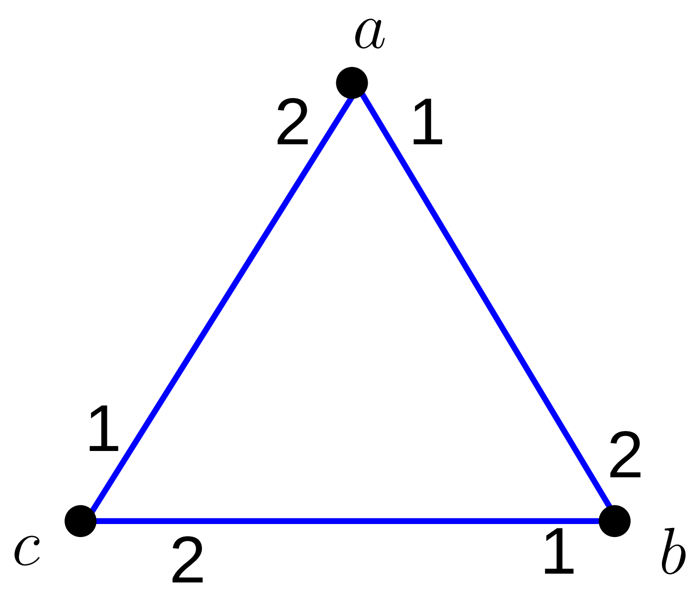
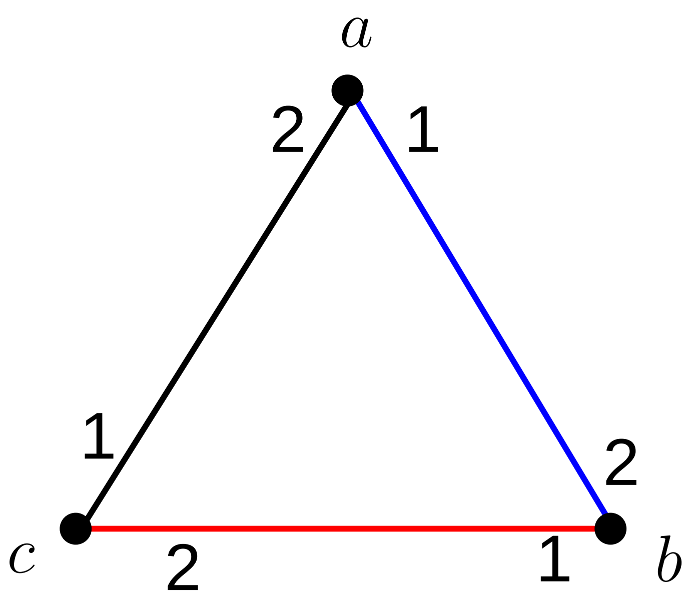


In general graphs

- In fact, this instance has no popular matching either.



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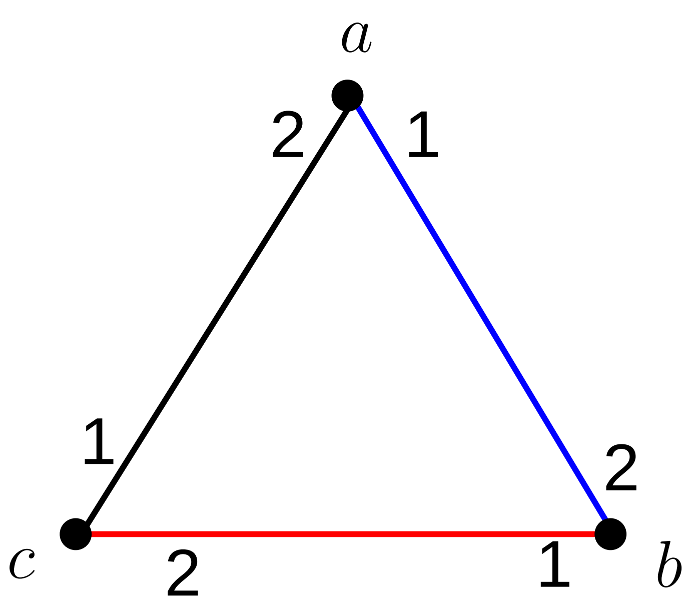

- We have $M_1 \prec M_2 \prec M_3 \prec M_1$ here, where $M_1 = \{(a,b)\}$, $M_2 = \{(b,c)\}$, and $M_3 = \{(a,c)\}$.

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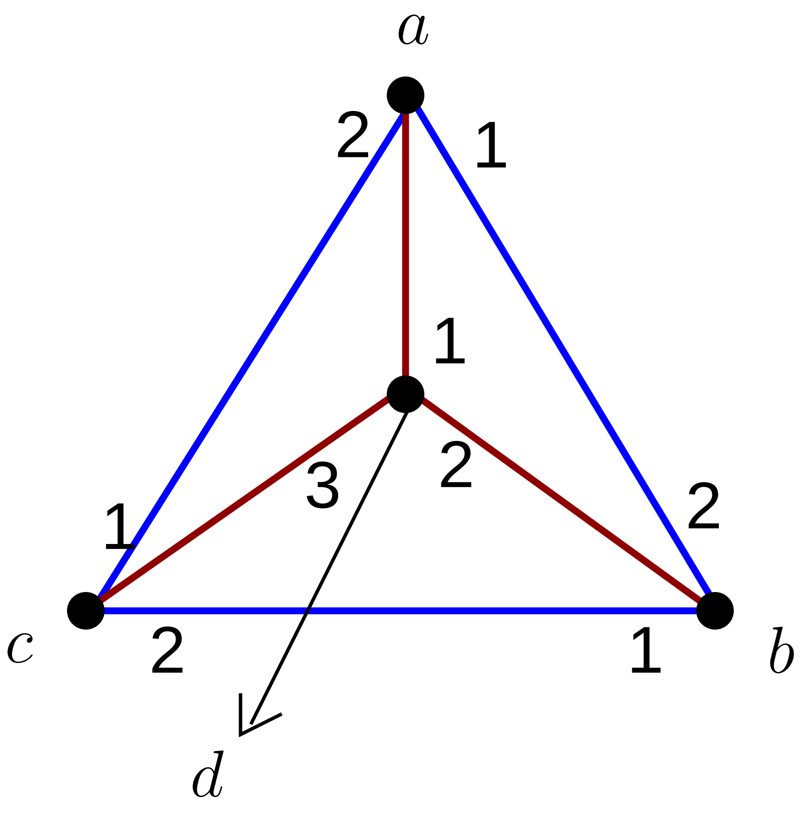

- d is the least preferred neighbor for a, b, c.

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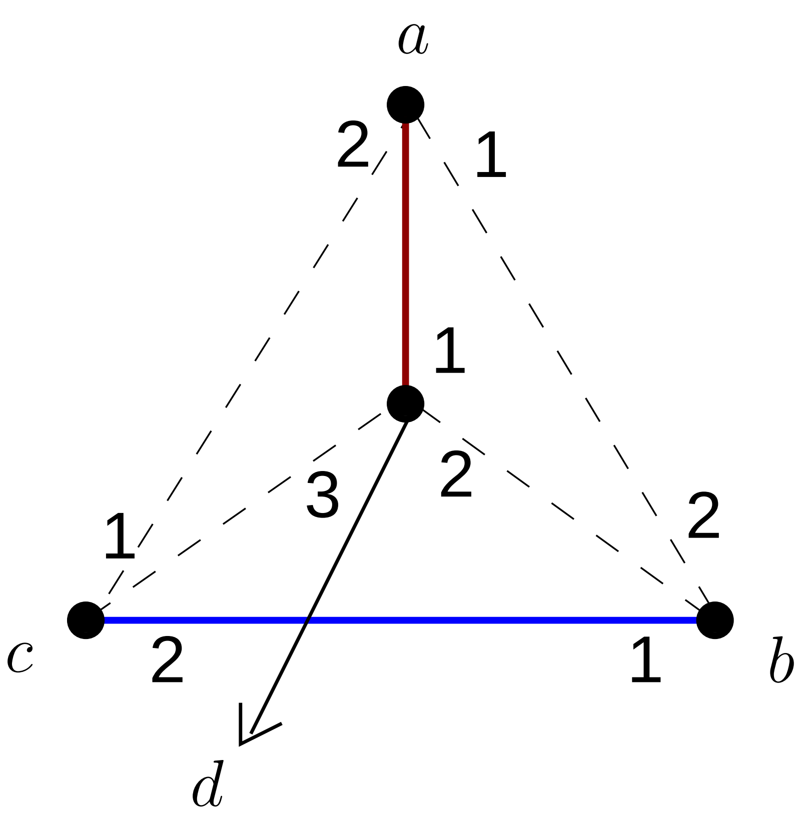

- $\{(a, d), (b, c)\}$ is popular.

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- There is always a matching M in G such that $u(M)$ is $O(\log n)$. (Huang and K 2013)
- Computing a least unpopularity factor matching in G is NP-hard.
- *Open problem*: the complexity of the popular matching problem in G.

Thank you!