Fair Stable Matchings

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Why study fair stable matchings?

In spite of the fact that an instance can have an *exponential* number of stable matchings, Gale-Shapley's algorithm outputs the man-optimal and woman-optimal stable matchings of an instance only – which is *really* good for one side of the matching but very *bad* for the other side.

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- not all recent work
- highlight nice results with some extra insight

Linear Programming Brings Marital Bliss
by J.H. Vande Vate

Operations Research Letters, 1989

Geometry of Fractional Stable Matchings and its Applications
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*Study classic SM and SR problems using a polyhedral approach

*Contains interesting results from a fairness perspective.

A stable marriage (SM) instance I:

- n men: $m_1, m_2, \ldots, m_n$
- n women: $w_1, w_2, \ldots, w_n$
- Each person has a preference list that is linear and complete.

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A *stable matching* μ is a perfect matching with no *blocking pairs* – i.e. an (m, w) such that

$$\mu(m) <_m w \text{ and } \mu(w) <_w m.$$

Now, we can also represent μ as an $n \times n$ permutation matrix X_μ. For example, if $\mu = \{(m_1, w_2), (m_2, w_1), (m_3, w_3)\}$,

$$X_\mu = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The (i, j)th entry of $X_\mu = 1$ iff $(m_i, w_j) \in \mu$.

A fractional stable matching X^f of I is a convex combination of some of the stable matchings of I. That is,

$$X^f = \sum_{i=1}^{r} \lambda_i X_{\mu_i}$$

where $0 < \lambda_i \leq 1$, $\sum_{i=1}^{r} \lambda_i = 1$ and μ_i is a stable matching of I.

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$$\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

*Clearly, X^f is an $n \times n$ doubly stochastic matrix.

Now consider an arbitrary $n \times n$ doubly stochastic matrix X.

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

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When does X represent a stable matching or a fractional stable matching of I?

In 1989, Vande Vate came up with an LP formulation for the stable marriage problem and introduced a set of inequalities.

We say that X satisfies the blocking inequalities of I if $\forall i, j$

$$X_{i,j} + \sum_{k: w_k <_{m_i} w_j} X_{i,k} + \sum_{k: m_k <_{w_j} m_i} X_{k,j} \leq 1.$$

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- When μ is a stable matching of I, the permutation matrix X_μ clearly satisfies the blocking inequalities of I.
- Thus, if X^f is a fractional stable matching, X^f also satisfies the blocking inequalities of I.

But what about the converse?

Theorem: (Birkhoff-von Neumann) Let D be an $n \times n$ doubly stochastic matrix. Then D is the convex combination of r permutation matrices. Moreover, $r \leq n^2$.

A BvN-like Theorem for Stable Matchings

Theorem: (VV '89, T&S '98) Let I be an SM instance with n men and n women. Let X be $n \times n$ doubly stochastic matrix that satisfies the blocking inequalities of I. Then X is the convex combination of r permutation matrices each of which satisfies the blocking inequalities of I. Moreover, $r \leq n^2$.

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- Every $n \times n$ doubly stochastic matrix X that satisfies the blocking inequalities of I *is* a fractional stable matching.
- Every fractional stable matching of I has a *concise representation* in terms of the stable matchings of I.

An aside: Sampling Stable Matchings Uniformly at Random

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- **Issue:** I may have an exponential number of stable matchings

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Approach 2: Use the Markov Chain Monte Carlo method.

- Effectively a random walk on the distributive lattice of the stable matchings of I.
- **Issue:** Bhatnagar et al. (2008) showed that *even when the preference lists are generated in a restricted way*, the mixing time of the Markov Chain can take exponential time.
  - The distributive lattice can look like an hour glass.

A new suggestion: Mimic the uniform distribution

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That is, if I has N stable matchings then

$$P_0 = \sum_{i=1}^{N} \frac{1}{N} X_{\mu_i}$$

where μ_i, $i = 1, \ldots, N$ are the stable matchings of I.

According to the BvN-like Theorem for stable matchings,

$$P_0 = \sum_{i=1}^{r} \lambda_i X_{\mu_i}$$

where $\lambda_i > 0$, $\sum_{i=1}^{r} \lambda_i = 1$ and $r \leq n^2$.

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That is, we can run a lottery using $r \leq n^2$ stable matchings of I and have the same expected result as the uniform distribution!

So what are these stable matchings, and what probability distribution should we assign to them?

Teo & Sethuraman's decomposition of P_0

Suppose SM instance I has N stable matchings.

For each man m, collect his partners from the N stable matchings and arrange them from his most preferred to least preferred woman. Let $p_i(m)$ denote the ith woman in this sorted list. For $i = 1, \ldots, N$, let

$$\alpha_i = \{(m, p_i(m)), m \in M\}.$$

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Theorem:(T&S) For $i = 1, \ldots, N$, α_i is a stable matching of I.

Example

	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9	μ_{10}
m_1	w_1	w_2	w_1	w_2	w_2	w_3	w_3	w_4	w_3	w_4
m_2	w_2	w_1	w_2	w_1	w_4	w_1	w_4	w_3	w_4	w_3
m_3	w_3	w_3	w_4	w_4	w_1	w_4	w_1	w_1	w_2	w_2
m_4	w_4	w_4	w_3	w_3	w_3	w_2	w_2	w_2	w_1	w_1

Example

	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9	μ_{10}
m_1	w_1	w_2	w_1	w_2	w_2	w_3	w_3	w_4	w_3	w_4
m_2	w_2	w_1	w_2	w_1	w_4	w_1	w_4	w_3	w_4	w_3
m_3	w_3	w_3	w_4	w_4	w_1	w_4	w_1	w_1	w_2	w_2
m_4	w_4	w_4	w_3	w_3	w_3	w_2	w_2	w_2	w_1	w_1

After sorting each man's partners,

	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}
m_1	w_1	w_1	w_2	w_2	w_2	w_3	w_3	w_3	w_4	w_4
m_2	w_2	w_2	w_1	w_1	w_1	w_4	w_4	w_4	w_3	w_3
m_3	w_3	w_3	w_4	w_4	w_4	w_1	w_1	w_1	w_2	w_2
m_4	w_4	w_4	w_3	w_3	w_3	w_2	w_2	w_2	w_1	w_1

Observations on the α_i's:

- α_1 is the man-optimal stable matching.
- α_N is the woman-optimal stable matching.
- $\alpha_1 \geq_m \alpha_2 \geq_m \alpha_3 \ldots \geq_m \alpha_N$ for each man m.
  - the α_i's form a chain in $\mathcal{L}(I)$.
  - hence, there are at most n^2 *distinct* α_i's. WHY?

The decomposition:

Let $S = \{\mu : \mu = \alpha_i, i \in \{1, 2, \ldots, N\}\}$. For each $\mu \in S$, let

$$\pi(\mu) = |\{i : \alpha_i = \mu\}|/N.$$

It's not difficult to see that $P_0 = \sum_{\mu \in S} \pi(\mu) X_\mu$.

Example cont'd

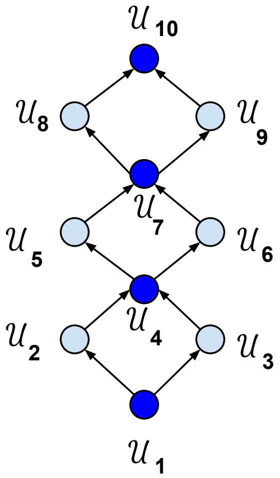
	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}
m_1	w_1	w_1	w_2	w_2	w_2	w_3	w_3	w_3	w_4	w_4
m_2	w_2	w_2	w_1	w_1	w_1	w_4	w_4	w_4	w_3	w_3
m_3	w_3	w_3	w_4	w_4	w_4	w_1	w_1	w_1	w_2	w_2
m_4	w_4	w_4	w_3	w_3	w_3	w_2	w_2	w_2	w_1	w_1

Example cont'd

	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}
m_1	w_1	w_1	w_2	w_2	w_2	w_3	w_3	w_3	w_4	w_4
m_2	w_2	w_2	w_1	w_1	w_1	w_4	w_4	w_4	w_3	w_3
m_3	w_3	w_3	w_4	w_4	w_4	w_1	w_1	w_1	w_2	w_2
m_4	w_4	w_4	w_3	w_3	w_3	w_2	w_2	w_2	w_1	w_1

$$\mu_1 = \{(m_1, w_1), (m_2, w_2), (m_3, w_3), (m_4, w_4)\}$$
$$\mu_4 = \{(m_1, w_2), (m_2, w_1), (m_3, w_4), (m_4, w_3)\}$$
$$\mu_7 = \{(m_1, w_3), (m_2, w_4), (m_3, w_1), (m_4, w_2)\}$$
$$\mu_{10} = \{(m_1, w_4), (m_2, w_3), (m_3, w_2), (m_4, w_1)\}$$

So $P_0 = \frac{2}{10}X_{\mu_1} + \frac{3}{10}X_{\mu_4} + \frac{3}{10}X_{\mu_7} + \frac{2}{10}X_{\mu_{10}}$.



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- It can be used to mimic *any* probability distribution on the set of stable matchings (incl. the uniform distribution) *in a concise way*.
- T & S's decomposition of X^f makes use of a set of stable matchings that form a chain. It is the *only* decomposition that forms a chain.

Geometry of Fractional Stable Matchings and its Applications by C.P. Teo and J. Sethuraman

Mathematics of Operations Research, 1998

Understanding the Generalized Median Stable Matchings by C. Cheng

Algorithmica, 2010

The counterpart of the α_i's

Suppose SM instance I has N stable matchings.

For each woman w, collect her partners from the N stable matchings and arrange them from her most preferred to least preferred man. Let $p_i(w)$ denote the ith man in this sorted list. For $i = 1, \ldots, N$, let

$$\beta_i = \{(p_i(w), w), w \in W\}.$$

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Theorem: (T&S) For $i = 1, \ldots, N$, $\alpha_i = \beta_{N-i+1}$.

[Fleiner (2003) and Klaus and Klijn (2006) proved the existence of the α_i's using different approaches.]

Example cont'd

	α_1 β_{10}	α_2 β_9	α_3 β_8	α_4 β_7	α_5 β_6	α_6 β_5	α_7 β_4	α_8 β_3	α_9 β_2	α_{10} β_1
m_1	w_1	w_1	w_2	w_2	w_2	w_3	w_3	w_3	w_4	w_4
m_2	w_2	w_2	w_1	w_1	w_1	w_4	w_4	w_4	w_3	w_3
m_3	w_3	w_3	w_4	w_4	w_4	w_1	w_1	w_1	w_2	w_2
m_4	w_4	w_4	w_3	w_3	w_3	w_2	w_2	w_2	w_1	w_1

Example cont'd

	α_1 β_{10}	α_2 β_9	α_3 β_8	α_4 β_7	α_5 β_6	α_6 β_5	α_7 β_4	α_8 β_3	α_9 β_2	α_{10} β_1
m_1	w_1	w_1	w_2	w_2	w_2	w_3	w_3	w_3	w_4	w_4
m_2	w_2	w_2	w_1	w_1	w_1	w_4	w_4	w_4	w_3	w_3
m_3	w_3	w_3	w_4	w_4	w_4	w_1	w_1	w_1	w_2	w_2
m_4	w_4	w_4	w_3	w_3	w_3	w_2	w_2	w_2	w_1	w_1

Thus, every participant in the middle α_i's is matched to his/her (lower or upper) median stable partner!

Define the *median stable matching* of I as

- $\alpha_{(N+1)/2}$ when N is odd and
- $\alpha_{N/2}$ and $\alpha_{N/2+1}$ when N is even.

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Teo and Sethuraman asked the following question:

Q: What is the computational complexity of finding the median stable matching of an SM instance?

- Using the definition will require enumerating all the stable matchings of the instance – and this can take exponential time.

The medians of a distributive lattice

Def: Let G be a connected graph. A vertex v of G is a *median* of G if its total (or average) distance from all other vertices of G is the least.

In the 1960's, Barbut initiated the study of medians of distributive lattices by using the covering graphs of these lattices. He showed that they behaved "nicely." This leads to an intriguing question:

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Q: What is the relationship between the medians of $\mathcal{L}(I)$ and the median stable matchings of I?

Contributions

Theorem: (Cheng, Nemoto 2000) [Characterization] For each rotation ρ in $\mathcal{P}_{\mathcal{L}(I)}$, let n_ρ denote the number of closed subsets that contain ρ. Then,

$$\alpha_i \text{ corresponds to } \{\rho : n_\rho \geq N - i + 1\}.$$

In particular, when N is odd, $\alpha_{(N+1)/2}$ corresponds to

$$\{\rho : \ \rho \text{ appeared in majority of the closed subsets}\}.$$

By relating the new characterization with the results of Barbut, we have the following:

Theorem: (Cheng) [Fairness] Suppose I has N stable matchings.

a. When N is odd, $\alpha_{(N+1)/2}$ is the unique median vertex of $\mathcal{L}(I)$.
b. When N is even, a stable matching μ is a median vertex of $\mathcal{L}(I)$ if and only if $\alpha_{N/2} \preceq \mu \preceq \alpha_{N/2+1}$.

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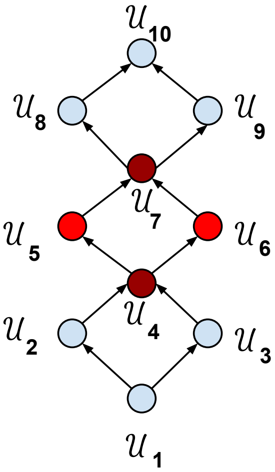
Thus, SM instances have stable matchings that are fair "locally" and "globally". We call this the *local/global median phenomenon*.



For the instance I, $\alpha_5 = \mu_4$, $\alpha_6 = \mu_7$ and every stable matching μ such that $\mu_4 \preceq \mu \preceq \mu_7$ is a median of $\mathcal{L}(I)$.

Theorem: (Cheng) [Complexity] When i is $O(\log n)$, α_i can be computed efficiently. But in general, it is #P-hard.

- If there is an efficient algorithm for computing the median stable matching of an SM instance, then there is an efficient algorithm for counting the number of stable matchings of an SM instance. But the latter is #P-complete.

My approach: use poset representation of stable matchings

An aside: Posets and Distributive Lattices

posets $\Leftrightarrow$ distributive lattices

Let $\mathcal{P} = (P, \leq)$ be a poset. A subset P' is a *closed subset (also down-set or order-ideal)* of P if whenever $y \in P'$ then so is $x \in P'$ whenever $x < y$.

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$\Rightarrow$ **(Folklore)** Let $CS(\mathcal{P})$ consist of the closed subsets of $\mathcal{P}$. Then $(CS(\mathcal{P}), \subseteq)$ is a distributive lattice.

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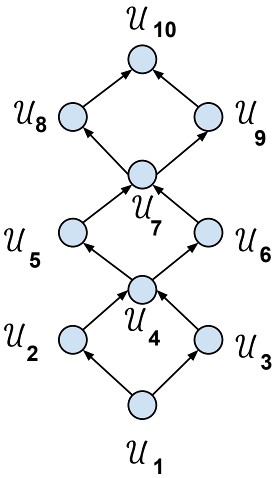
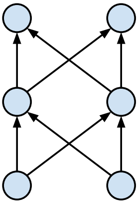
⇐ **(Birkhoff)** For every distributive lattice $\mathcal{L}$, there is a poset $\mathcal{P}_{\mathcal{L}}$ so that $(CS(\mathcal{P}_{\mathcal{L}}), \subseteq)$ is order isomorphic to $\mathcal{L}$.

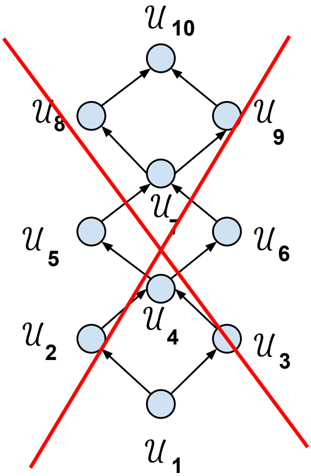
Every distributive lattice $\mathcal{L}$ can be encoded by a *poset* $\mathcal{P}_{\mathcal{L}}$.

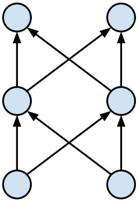




The poset $\mathcal{P}_{\mathcal{L}}$ associated with the distributive lattice $\mathcal{L}$ is shown on the right.

For stable matchings: the poset can be constructed directly from the instance.





stable marriage instances $\Leftrightarrow$ posets

$\Rightarrow$ **(Irving et al.)** Suppose I in an SM instance with n men and n women with $\mathcal{L}(I)$ as the its distributive lattice of stable matchings.

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- $\mathcal{P}_{\mathcal{L}(I)}$ can be derived directly from the preference lists of the participants. It's called the rotation poset of I.
- It has at most $O(n^2)$ elements called *rotations*, and it can be constructed in $O(n^2)$ time.

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- It has at most $O(n^2)$ elements called *rotations*, and it can be constructed in $O(n^2)$ time.

$\Leftarrow$ **(Blair, Gusfield et al.)** For every poset $\mathcal{P}$, there is an SM instance $I_{\mathcal{P}}$ whose rotation poset is order-isomorphic to $\mathcal{P}$. Moreover, its size is $O(poly(|\mathcal{P}|)$.

Q: Which closed subset of $\mathcal{P}_{\mathcal{L}(I)}$ corresponds to α_i for $i = 1, \ldots, N$?

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* The new characterization answered this question.

Fact No. 2: Every SM instance has a stable matching that is both locally median and globally median. Unfortunately, for general instances, it is #P-hard to compute such a stable matching.

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- When the rotation poset associated with the instance is series-parallel, an interval order or 2-dimensional, computing a median stable matching can be done efficiently.
- The local/global median phenomenon applies to an arbitrary collection of stable matchings.

Two Extensions

Stable Roommates Matchings, Mirror Posets, Median Graphs, and the Local/Global Median Phenomenon in Stable Matchings
by C. Cheng and A. Lin

SIAM Journal of Discrete Math, 2011

The center stable matchings and the centers of cover graphs of distributive lattices
by C. Cheng, E. McDermid and I. Suzuki

ICALP 2011
[A journal version of the paper is under submission.]

Extension 1. Stable Roommates

T & S noted that solvable Stable Roommates (SR) instances also have median stable matchings.

Are the median stable matchings also globally median?

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Are the median stable matchings also globally median?

State of knowledge at that time:

SM instances	$\Leftrightarrow$	**posets**	$\Leftrightarrow$	**distributive lattices**
SR instances	$\Rightarrow$	**mirror posets**		**??**

Cheng and Lin showed

- Like SM instances, SR instances also had “dualities”:

  SR instances $\Leftrightarrow$ **mirror posets** $\Leftrightarrow$ **median graphs**

- A median stable matching of a solvable SR instance is also a median vertex of its median graph.

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Fact No. 3a: The local/global median phenomenon extends to solvable SR instances.

Extension 2. Center Stable Matchings

Let G be a connected graph.

Def: A *center* of G is a node whose maximum distance from another node of G is the least.

Def: Given an SM instance I, *center stable matching* of I is a center of the cover graph of $\mathcal{L}(I)$.

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Def: Given an SM instance I, *center stable matching* of I is a center of the cover graph of $\mathcal{L}(I)$.

Like a median stable matching of I, a center stable matching of I is "fair" because it is a good representative of I's stable matchings.

Q: What is the computational complexity of computing a center stable matching of $\mathcal{L}(I)$?

Cheng, McDermid & Suzuki showed

- A center stable matching of an SM instance I can be computed in polynomial time.
- A characterization of *all* the center stable matchings of I.

  - Some center stable matchings are the middle nodes of a longest chain of $\mathcal{L}(I)$ but the converse is not true.

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Fact No. 3b: A center stable matching is another globally fair stable matching. It can be computed efficiently.

On the Stable Matchings that can be Reached When the Agents Go Marching in One by One
by C. Cheng

under submission, 2014

In the Gale-Shapley Algorithm,

- only one group can make a proposal
- the output favors the proposing group

A common question I get from students: *Is there are algorithm where both men and women propose, and will that result in a less biased output?*

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One possibility: *Ma's Random Order Mechanism (proposed in 1996), a sequential version of the Gale-Shapley algorithm.*

How the Random Order Mechanism (ROM) works:

- Start with a random permutation of the participants π.
- At the beginning of each iteration i,
  - there is a stable matching μ_{i-1} for the participants in $\pi(1 \cdots i-1)$.
  - $\pi(i)$ marches in and starts proposing to the person he or she prefers the most among those in the room.
  - a GS-algorithm-like step ensues where the individuals on the side of μ_i proposing.
  - at the end of the iteration, there is a stable matching μ_i for the i participants.

How the Random Order Mechanism (ROM) works:

Let $\pi = \mathbf{m_1}, w_2, w_1, m_2, w_3, m_3, m_4, w_4$.

m_1:	w_1	w_2	w_3	w_4	w_1:	m_4	m_3	m_2	m_1
m_2:	w_2	w_1	w_4	w_3	w_2:	m_3	m_4	m_1	m_2
m_3:	w_3	w_4	w_1	w_2	w_3:	m_2	m_1	m_4	m_3
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m_2:	w_2	w_1	w_4	w_3	w_2:	m_3	m_4	$\underline{m_1}$	m_2
m_3:	w_3	w_4	w_1	w_2	w_3:	m_2	m_1	m_4	m_3
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Skipping ahead...

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m_3:	$\underline{w_3}$	w_4	w_1	w_2	w_3:	m_2	m_1	m_4	$\underline{m_3}$
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m_3:	w_3	w_4	w_1	w_2	w_3:	m_2	m_1	$\underline{m_4}$	m_3
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m_3:	w_3	w_4	$\underline{w_1}$	w_2	w_3:	m_2	m_1	$\underline{m_4}$	m_3
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m_3:	w_3	w_4	$\underline{w_1}$	w_2	w_3:	$\underline{m_2}$	m_1	m_4	m_3
m_4:	w_4	w_3	$\underline{w_2}$	w_1	w_4:	$\underline{m_1}$	m_2	m_3	m_4

ROM can reach stable matchings different from the man-optimal and woman-optimal stable matchings!

Facts about ROM:(Ma (1996), Blum et al. (1997), Cechalárová (2002))

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- ROM will output a stable matching in $O(n^3)$ time.
- ROM can simulate the Gale-Shapley algorithm.
  - when π consist of all men followed by all women, ROM(π) will output the woman-optimal SM, etc.
- ROM will always match the last person in π to his/her best stable partner.

  Consequence: *If no agent in μ is matched to his/her best stable partner, ROM will never output μ.*

Assume the permutation π, the input to ROM, was chosen uniformly at random.

- Klaus and Klijn argued that this is a *procedurally fair* mechanism for generating a stable matching.

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- Some natural questions to ask –

  What is the *probability distribution induced by ROM* on the set of stable matchings?

  What is the *support* of this probability distribution?

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- Some natural questions to ask –

  What is the *probability distribution induced by ROM* on the set of stable matchings?

  What is the *support* of this probability distribution?

Call μ *ROM-reachable* if there is a permutation π of the agents so that ROM(π) outputs μ.

Which stable matchings are ROM-reachable?

Results:

- Given a stable matching μ, determining if μ is ROM-reachable is NP-complete.
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  - it is enough to check *m* permutations where *m* is the number of participants.

Results:

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  - the difficulty lies in the unstable partners
- Determining if ROM can output a non-trivial stable matching can be done in polynomial time.
  - it is enough to check m permutations where m is the number of participants.
- Additional results on "strongly ROM-reachable" and "extreme" stable matchings.

Fact No. 4: The Random Order Mechanism, a sequential version of the Gale-Shapley algorithm, can output other kinds of stable matchings. Determining if a given stable matching is ROM-reachable, however, is NP-complete.

Postscript

- Some open questions –
  (i) Can a center stable matching of a solvable SR instance be computed efficiently?
  (ii) How is ROM related to Random Serial Dictatorship when the set of objects are stable matchings?

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- Some open questions –
  (i) Can a center stable matching of a solvable SR instance be computed efficiently?
  (ii) How is ROM related to Random Serial Dictatorship when the set of objects are stable matchings?
- Studying the fairness issue in stable matchings has led to some very interesting structural results. Can they be applied to other objects that form a distributive lattice?
  - domino tilings of a polygon
  - the matchings of a connected bipartite planar graph
  - independent sets in a bipartite graph
  - alternating sign matrices