One dimensional mechanism design

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prior-free mechanism design:

the central tradeoff

- efficiency
- incentive compatibility
- fairness

Hurwicz 1972, Gibbard/Satterthwaite 1974, Green/Laffont 1979, $\cdots$

famous exceptions

- assignment with property rights (Ma 1994, Papai 2000)

- assignment by random priority (Abdulkadiroglu/Sonmez 1999)

- random matching with dichotomous preferences (Bogomolnaia/Moulin 2004)

- regular matching falls short

the single-peaked exception

- (*very well known*) voting over a line of candidates under **single-peaked** (convex) preferences: the median peak is the Condorcet winner (Black 1948), defining an incentive-compatible voting rule (Dummett and Farquharson 1961, Pattanaik 1974); the generalized median rules (Moulin 1980) preserve this property

- (*less well known*) dividing a single non disposable commodity (workload) under convex private preferences: the uniform division rule (Sprumont 1991, Barbera/Jackson/Neme 1997, $\cdots$)

- (variants) balancing one dimensional demand and supply (Klaus/Peters/Storcken 1998); under bipartite constraints (Bochet et al. 2012) *critical features*

$\longrightarrow$ one dimensional individual allocations

$\longrightarrow$ single-peaked private preferences over own allocation

$\longrightarrow$ convex set of allocation profiles

many more examples share these features

individual allocations may represent different commodities production chain with two teams: $N = L \cup R$

and substitute team members

$$\sum_{i \in L} x_i = \sum_{j \in R} y_j$$

e.g., the L-team extracts the input (raw material, customers' orders) which is processed by the R-team

if R contains a single "manager" we have a moneyless principal-agent problem

intuitively: vote between teams followed by a division inside each team production chain with three teams $N = L \cup C \cup R$

and complementary team members

$$x + y + z = 100$$

$$x_i = \lambda_i x \text{ all } i \in L \ ; \ \ y_j = \mu_j y \text{ all } j \in C \ ; \ \ z_k = \upsilon_k z \text{ all } k \in R$$

intuitively: vote inside the teams and a division problem between teams workload division under bilateral constraints

$w^{kl} =$ total work at time k and location l

exogenous constraints: $\sum_l w^{kl} = W^k$, $\sum_k w^{kl} = W_l$

contractor i cares about total volume $x_i = \sum_{k,l} w_i^{kl}$

and faces various linear constraints like $w_i^{kl} = 0$, $w_i^{kl} + w_i^{kl'} \leq C$ etc..

$\longrightarrow$ **in all these examples the tradeoff disappears:**

we can construct simple mechanisms

efficient

incentive compatible (strategyproof)

and fair (symmetric treatment of agents; envy-freeness; individual guarantees) *general model*

N the relevant agents

allocation profile $x = (x_i)_{i \in N} \in \mathbb{R}^N$

feasibility constraints: $x \in Z$ *closed and convex* in $\mathbb{R}^N$

Z_i : projection of Z on the i-th coordinate

agent i's preferences $\succeq_i$ are single-peaked over Z_i with peak p_i

direct revelation mechanism

$$F: \ (\succeq_i)_{i \in N} \to x \in Z$$

peak-only revelation mechanism (much easier to implement)

$$f : p = (p_i)_{i \in N} \to x = f(p) \in Z$$

such that

$$F(\succeq_i;_{i \in N}) = f(p_i; i \in N)$$

- efficiency (**EFF**) i.e., *Pareto optimality*

- incentive compatibility: *StrategyProofness* (**SP**), *GroupStrategyProofness* (**GSP**), or *StrongGroupStrategyProofness* (**SGSP**)

- *Continuity* (**CONT**): F (resp. f) is continuous for the “right” topology on preferences (resp. peaks) *note*: in this general setting, SP does not imply GSP (Barbera et al. 2014)

note: SP and Continuity together imply *peak-only*

A folk proposition

a **fixed priority rule** meets EFF, SGSP, and CONT

agent 1 is guaranteed her peak

conditional on this, agent 2 is guaranteed his best feasible allocation

conditional on this, agent 3 is guaranteed his best feasible allocation

...

note: only Continuity requires the convexity of Z

Fairness Axioms

- *Symmetry* (**SYM**): $F((\succeq_{\sigma(i)})_{i\in N}) = (x_{\sigma(i)})_{i\in N}$ *if the permutation* $\sigma : N \longrightarrow N$ *leaves* Z *invariant*

- *Envy-Freeness* **(EF)**: *if permuting* i *and* $j : N \longrightarrow N$ *leaves* Z *invariant then* $x_i \succeq_i x_j$

- ω*-Guarantee* **(**ω**-G)**: $x_i \succeq_i \omega_i$ *for all* i, *where* $\omega \in Z$

an allocation $\omega \in Z$ is *symmetric* if $\omega^\sigma = \omega$ for every σ leaving Z invariant,

Main Theorem

For any convex closed problem (N, Z)*, and any symmetric allocation* $\omega \in Z$*, there exists at least one peak-only mechanism* f *that is Efficient, Symmetric, Envy-Free, Guarantees-*ω*, Continuous, and SGSP*

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the proof of the main theorem is constructive

Step 1: for any $\omega \in Z$, symmetric or not, we define a peak-only mechanism f^{ω} meeting EFF, ω-G, CONT, and SGSP

$\longrightarrow$ CONT is the hardest to prove

Step 2: if ω is a then f^{ω} is Symmetric and Envy-Free as well notation in $\mathbb{R}^N$

$a \to a^* \in \mathbb{R}^n$ by rearranging the coordinates of a increasingly

the leximin ordering applies the lexicographic ordering to a^*:

$$a \succeq_{\text{leximin}} b \iff a^* \succeq_{\text{lexicog}} b^*$$

this is a complete symmetric ordering of $\mathbb{R}^N$ that is *discontinuous*

its maximum over a compact set may not be unique

but over a closed compact set it is unique notation: $[a,b] = [a \wedge b, a \vee b]$ and $|a| = (|a_i|)_{i \in N}$

define the canonical **leximin rule** f^ω

$$f^\omega(p) = x \overset{def}{\Longleftrightarrow} \{\, x \in Z \cap [\omega, p] \text{ and}$$

$$|x - \omega| \succeq_{\text{leximin}} |y - \omega| \text{ for all } y \in Z \cap [\omega, p]\}$$

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a key subclass of problems:

the problem (N, Z) is *anonymous*

if Z is symmetric in **all** permutations the affine span $H[Z]$ of an anonymous convex set Z is one of three types

- $H[Z]$ is the diagonal $\mathbf{\Delta}$ of $\mathbb{R}^N$: Z is a *voting problem*
- $H[Z]$ is parallel to $\mathbf{\Delta}^\perp = \{\sum_N x_i = 0\}$: Z is a *division problem*
- $H[Z] = \mathbb{R}^N$ a new class of problems

Case 1: $H[Z] = \Delta$: *then* Z *is a voting problem*

the $(n-1)$-dimensional family of *generalized median rules* meets EFF, SYM, CONT and SGSP (is characterized by EFF + SYM + SP)

$$f(p) = \text{median}\{p_i, i \in N; \alpha_k, 1 \leq k \leq n-1\}$$

f^ω is the rule most biased toward the status quo ω: $\alpha_k = \omega$ for all k. It takes the *unanimous* voters to move away from the status quo

Case 2: $H[Z]$ *parallel to* $\Delta^{\perp}$: then Z involves dividing a single commodity

$\longrightarrow$ if Z is the "simplex" division problem $Z = \{x \geq 0 , \sum_N x_i = 1\}$ then ω is equal split and f^{ω} is the *uniform rule* (Sprumont 1991)

$$f_i^{\omega}(p) = \min\{\lambda, p_i\} \text{ if } \sum_N p_i \geq 1 \text{ ; } f_i^{\omega}(p) = \max\{\lambda, p_i\} \text{ if } \sum_N p_i \leq 1$$

$\longrightarrow$ if Z is the supply-demand problem $Z = \{\sum_N x_i = 0\}$ then $\omega = 0$ and f^0 *serves the short side while rationing uniformly the long side* *classic results*

$\longrightarrow$ in the simplex division the uniform rationing rule f^{ω} is the **unique** mechanism meeting EFF, SYM and SP (Ching 1994)

$\longrightarrow$ in the supply-demand problem the uniform rationing rule f^{0} is the **unique** mechanism meeting EFF, SYM, SP, and guaranteeing voluntary participation general anonymous division problem: $Z = \{\sum_N x_i = \beta\} \cap C$

example: dividing shares in a joint venture

$$x_N = 100$$

$$x_S \geq 51 \text{ if } |S| \geq \frac{2n}{3} \text{ legal constraint}$$

$$x_S \leq 66 \text{ if } |S| \leq \frac{n}{2} \text{ power balance constraint}$$

etc..

the uniqueness result generalizes

the only symmetric feasible allocation is $\omega_i = \frac{\beta}{n}$ for all i

Proposition

In an anonymous (convex) division problem $Z = \{\sum_N x_i = \beta\} \cap C$ the rule f^ω is characterized by EFF, SYM, CONT and SGSP

$\longrightarrow$ *conjecture:* SP suffices instead of SGSP

Case 3: $H[Z] = \mathbb{R}^N$

example: the *bounded variance* problem: $Z = \{\sum_N x_i^2 \leq 1\}$

with respect to the benchmark allocation $x_i = 0$ for all i, the system can accomodate adjustments with limited variance

$\longrightarrow$ superficially identical to the division of one unit by the change of variables $p_i \rightarrow \widetilde{p}_i = p_i^2$

$\longrightarrow$ in fact a host of mechanisms meet our four axioms

Figure 1 illustrates the case $n = 2$

ω can be anywhere on the diagonal of the ellipse about the convexity assumption

$\longrightarrow$ convexity of Z is not necessary in the main result

mechanisms with the announced properties exist for certain non convex feasible sets Z

Figure 2

but for some non convex feasible sets Z even EFF, SP, and CONT are incompatible

Figure 3: a non convex Z with no Efficient Continuous and Strategyproof mechanism

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in general many other rules than f^{ω} meet our five axioms, even if we require that the welfare level of ω be guaranteed to all participants

$\Longrightarrow$ anonymous *division* problems are an exception example: production chain with two teams: $N = L \cup R$

and substitute team members

$$\sum_{i \in L} x_i = \sum_{j \in R} y_j$$

symmetries of Z: inside L and inside R

agents in L report $p_i \geq 0$

agents in R report $q_j \geq 0$

$\longrightarrow$ total workload $t(p, q)$

Strong GSP $\Longrightarrow$ L-agents share $y = t(p, q)$ by the uniform rule; so do the R-agents

Efficiency $\Longleftrightarrow$ $t(p, q) \in [p_L, q_R]$

the mechanism f^0 guarantees "Voluntary Work": everyone can opt out and do no work

$$t(p, q) = \min\{p_L, q_R\}$$

the short side gets its peak allocation

the long side is uniformly rationed, as in the supply-demand problem with Voluntary Trade

crucial difference: fixed roles

many other choices for $t(p, q)$ (failing Voluntary Work)

a large family of possible choices

$$t(p, q) = median\{p_L, q_R, \theta(p, q)\}$$

where $(p, q) \to \theta(p, q)$ is an anonymous and strategyproof voting rule

$p \rightarrow p^* : p^{*1} \leq p^{*2} \leq \cdots \leq p^{*l}$

$q \rightarrow q^* : q^{*1} \leq q^{*2} \leq \cdots \leq q^{*r}$

$$\theta(p,q) = \max_{k,k'} \{ \min\{lp^{*k}, rq^{*k'}, \alpha_{k,k'}\}\}$$

where $\alpha_{k,k'}$ are arbitrary constants weakly decreasing in k, k'

for instance

$\theta(p,q) = lp^{*1}$

$\theta(p,q) = rq^{*r}$

$\theta(p,q) = median\{(lp^{median}) \times 1, (rq^{median}) \times 2\}$

etc..

note: this explains why we have a huge number of good mechanisms in the bounded variance problem

Conclusion

unification of previous results in a more general model

an embarrassment of riches

in one-dimensional problems with convex feasible outcome sets, we can design many efficient, incentive compatible (in a strong sense) and fair mechanisms

additional requirements must be imposed to identify reasonably small new families of mechanisms

Thank You