One dimensional mechanism design

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May 2015

Abstract

If preferences are single-peaked, electing the best choice of the median voter is an efficient, strongly incentive compatible and fair mechanism (Black (1958), Dummett and Farquharson (1961)). Dividing a single non disposable commodity by the uniform rationing rule meets these three properties as well when preferences are private and single-peaked (Sprumont (1991)).

These are two instances of a general possibility result applying to any collective decision problem where individual allocations are one-dimensional, preferences are single-peaked (strictly convex), and feasible allocation profiles cover a closed convex set. The proof is constructive, by means of a mechanism equalizing gains in the leximin sense from an arbitrary benchmark allocation. In most problems there are many more mechanisms combining efficiency, incentive compatibility and fairness.

1 Introduction and the punchline

Single-peaked preferences played an important role in the birth of social choice theory and mechanism design. Black observed in 1948 that the majority relation is transitive when candidates are aligned and preferences are single-peaked ([10]): this result inspired Arrow to develop the social choice approach with arbitrary preferences. Dummett and Farquharson observed in 1961 that the median peak (i.e., the majority winner) defines an incentive compatible voting rule ([20]); they also conjectured that no voting rule is incentive compatible under general preferences, which was proven true twelve years later by Gibbard and by Satterthwaite ([23], [37]).

Two decades and many more impossibility theorems later, single-peaked preferences reappeared in the problem of allocating a single *non disposable* commodity (e.g., a workload) when the agregate demand may be above or below the amount to be divided. Developing Benassy's earlier observation ([9]) that uniform rationing of a single commodity prevents the strategic inflation of individual demands, Sprumont ([42]) characterized the *uniform rationing rule* by combining the three perennial goals of prior-free mechanism design: efficiency, strategyproofness, and fairness.

This striking "if and only if" result is almost alone of its kind in the literature on mechanisms to allocate private commodities (see Section 3). By contrast in the voting problem there are *many* efficient, strategyproof and fair voting rules under single-peaked preferences: they are the "generalized median" rules ([31]).

We generalize both models, voting and non disposable division, to a collective decision problem where each participant is only interested in a one-dimensional "personal" allocation, his/her preferences are single-peaked (strictly convex) over this allocation, and some abstract constraints limit the set of feasible allocation profiles. The latter set is a line in the voting model, and a simplex for the non disposable division model; in general it is any closed convex set.

The main result is that we can always design "good" allocations mechanisms, i. e., efficient, incentive-compatible (in the strong sense of group-strategyproofness) and fair. Loosely speaking, in convex economies where each agent consumes a single commodity, the mechanism designer hits no impossibility wall.

The proof constructs a canonical good mechanism with the help of the leximin ordering, an important concept in post-Rawls welfare economics (see Section 3). Recall that the welfare profile w beats profile w' for this ordering if the smallest coordinate is larger in w than in w', or when these are equal, if the second smallest coordinate is larger in w than in w', and so on. In our model we fix a benchmark allocation ω that is fair in the sense that it respects the symmetries of the set of feasible allocation profiles. Then we equalize, as much as permitted by feasibility, individual benefits away from ω in the direction of individual peaks: that is, the profile of benefits maximizes the leximin ordering. Despite the fact that the leximin ordering is not continuous, this maximum is uniquely defined.

The corresponding mechanism, in addition to meeting the three basic goals, is continuous in the profile of peaks. We call it the *uniform gains rule*, to stress its similarity with the uniform rationing rule. Indeed in the non disposable division problem the two rules coincide. The uniform gains rule remains the unique good mechanism in more general division problems where the sum of individual allocations is constant, and the additional feasibility constraints are symmetric across agents but otherwise arbitrary.

However the "constant sum" problems above are an exception: in other fully symmetric problems (invariant when we swap any two agents) we expect that the mechanism designer faces an embarrassment of riches, that is to say a host of good mechanisms. We noticed this above in the voting problem, where a generalized median rule is described by $n-1$ free parameters (n is the number of voters). It remains true in the new class of problems where the set of feasible allocations is of dimension n: there good mechanisms form a set of infinite dimension.

2 Overview of the results

After reviewing the relevant literature in the next Section, we define the model in Section 4. Given the set N of agents, a problem is simply a closed convex subset X of $\mathbb{R}^N$, the set of feasible allocation profiles. Agent i has single-peaked preferences over the projection X_i of X onto his coordinate. If X is a subset of the diagonal of $\mathbb{R}^N$ we have a voting problem. If the sum $\sum_N x_i$ is constant in X we have a generalized division problem. We also give examples where X is of dimension $n = |N|$.

Two familiar notions of incentive compatibility are defined in Section 5: strategyproofness (SP) prevents individual strategic misreport, while strong groupstrategyproofness (SGSP) rules out cordinated moves by a group of agents, and guarantees non bossiness to boot. Under single-peaked preferences we expect a strategyproof revelation mechanism to be also peak-only: it only elicits individual peak allocations and ignores preferences across the peak. This is true in our general model provided the mechanism if continuous in the reports: Lemma 1.

The well known *fixed priority* mechanisms are, as usual, both efficient and SGSP. Therefore the point of our Theorem is to achieve these properties together with fairness requirements: we define three such properties in Section 6. Symmetry (horizontal equity) says that the mechanism must respect the symmetries between agents: if a permutation σ of the agents leaves X invariant, then relabeling agents according to σ will simply permute their allocations. Next Envy Freeness: if X is invariant by permuting i and j then i weakly prefers her own allocation x_i to j's allocation x_j. Finally we may want to guarantee that each participant weakly benefits above a benchmark allocation ω in X, that is, each agent i weakly prefers her allocation x_i to ω_i. We call this the ω-Guarantee property. As long as ω respects the symmetries of X, it is compatible with the other two.

We state the Theorem in Section 7. Given any symmetric allocation ω in X, we define the *uniform-gains rule* f^ω selecting the allocation in X where the profile of gains from ω_i toward the peak p_i maximizes the leximin ordering. This peak-only direct revelation mechanism is efficient, SGSP, symmetric, non envious, continuous, and guarantees ω.

Naturally we wish to understand what other good mechanisms are available: by this we mean that they meet all properties above except perhaps ω-Guarantee. Sections 8,9 provide some answers.

In Section 8 we focus on fully symmetric problems: X is unchanged by any permutation of the agents. A non trivial symmetric convex set in $\mathbb{R}^N$ can only be of dimension 1, $n-1$ or n, therefore there are exactly three types of fully symmetric problems.

Voting problems are those where X is of dimension 1. The uniform gains rule f^ω is strongly biased in favor of the *status quo* outcome ω: in order to elect another outcome, *all* individual peaks must be to the right of ω (or all to its left), and then the rule selects the peak closest to ω (Proposition 1).[1]

When X is of dimension $n-1$ the sum $\sum_N x_i$ must be constant (because X is symmetric) and we interpret X as a generalized division problem, of which the non disposable division model is but one example. There is only one symmetric allocation ω, and the uniform gains rule f^ω is the *unique* good rule: Proposition 2. This result applies to a much larger class of problems than Sprumont's characterization of the same rule ([42], [17]), on the other hand it requires more properties: SGSP in lieu of SP, and Continuity.

If X is of dimension n we have a new type of allocation problems of which we provide some examples. Here the set of good mechanisms is of infinite dimension, except in the two-person case where it coincides with the one-dimensional family f^ω parametrized by ω: Proposition 3.

Finally when the set X of feasible allocations is not fully symmetric, we expect that the set of good mechanisms (respecting the partial symmetries of X) to explode. We illustrate this in Section 9 by means of a very simple three-person workload division problem: workers $i = 1, 2$ bring each some amount x_i of input, and worker 3 must process the total output; the feasibility constraint is $x_3 = x_1 + x_2$. Symmetry rules out discrimination between workers 1 and 2, but it imposes no restriction to the relative treatment of 3 vis-a-vis 1 and 2. We describe four quite different subfamilies of good mechanisms, opening a rich avenue for future research.

We collect in Section 10 the proofs of the Theorem and Propositions 2,3.

[1] It is the generalised median rule where the fixed ballots are $n-1$ copies of ω. See Subsection 8.1.

3 Related literature

There is a folk impossibility about the design of prior-free mechanisms, where incentive compatibility is the strong requirement of strategyproofness: in economies where agents consume two or more commodities, a strategyproof mechanism must be either inefficient, grossly unfair, or both. To mention only a few salient contributions to this theme: Hurwicz conjectured ([26]), then Zhou proved ([48]) that the strategyproof and efficient allocation of private goods cannot guarantee "Voluntary Trade" (everyone weakly improves upon his initial endowment ω_i: see the ω-Guarantee axiom in Section 6); it cannot treat agents symmetrically either ([40]). In abstract quasi-linear economies, no strategyproof mechanism can be efficient ([24]); the same is true in public good economies ([5]); and so on.

There are very few exceptions. In the assignment of indivisible objects the top-trading-cycle is characterized by efficiency, SP and Voluntary Trade. And the impossibility disappear when all agents have Leontief preferences ([22], [29]). Our results show that the impossibility easily disappears in economies where each agent consumes a unique commodity, possibly a different commodity for different agents.

After the Gibbard Satterthwaite theorem, a substantial literature on voting rules looked for restrictions to the domain of preferences eschewing the impossibility. The single-peaked domain, the first known example, was extended in a variety of ways. If outcomes are arranged on a tree, the Condorcet winner still defines a good voting rule ([19]). If outcomes are a product of lines, there is a natural extension of single-peakedness in which coordinate-wise majority still yields a strategyproof and symmetric rule, though efficiency is replaced by the much weaker *Unanimity* property[2] ([4], [8], [7]), another instance of the "no rule is perfect in dimension two or more". Trees and products of lines are special cases of abstract convex sets, where we have a general characterization of strategyproof rules ([34], [35]).

Powerful recent results, still in the voting context, provide an endogenous characterization of (a generalization of) single-peaked domains by the fact that we can find strategyproof peak-only voting rules that are symmetric and unanimous ([13], [15], [16]).

Following Sprumont's result, the non disposable division problem received much attention as well. The uniform rationing rule was adapted to the supply-demand problem from an initial endowment of the commodity ([28]). It was extended to a random rule distributing indivisible units ([36],

[2] Outcome x is elected if it is the peak of all voters.

[41]), and more recently to a bipartite allocation model ([12], [11], [14]). Interestingly, the uniform rationing rule can be viewed as a fair division method, and then axiomatized in a variety of ways without invoking its incentive compatibility properties: see for instance [38], [44], [45].

If we drop the fairness requirement, there is an infinite dimensional set of efficient and strategyproof division rules: [6], [33], [21]. By contrast in Proposition 3 we find an infinite dimensional set of rules sharing these properties and fair as well.

A good survey of the literature on strategyproof voting and non disposable division rules is [2].

In modern welfare economics the leximin ordering was introduced by Sen ([39]) as a tool to implement Rawls' egalitarian program. Maximizing this ordering is sometimes called *practical egalitarianism*, as it guarantees efficiency while deviating as little as possible from the ideal of full equality of welfares. This ordering was axiomatized first as a social welfare ordering ([25], [1]), then as an axiomatic bargaining solution ([27], [46], [18]).

4 The model and some examples

The finite set of relevant agents is N with cardinality n. An allocation profile is $x = (x_i)_{i \in N} \in \mathbb{R}^N$. The set of feasible allocations is a *closed* subset X of $\mathbb{R}^N$. The projection of X on the i-th coordinate captures agent i's feasible allocations; it is a closed set $X_i \subseteq \mathbb{R}$; the cartesian product of these sets is $X_N = \Pi_{i \in N} X_i$.

Agent i's preferences $\succeq_i$ are single-peaked over X_i if 1) there is some $p_i \in X_i$, the peak, that $\succeq_i$ ranks strictly above any other, and 2) $\succeq_i$increases strictly with x_i on $X_i \cap]-\infty, p_i]$ and decreases strictly on $X_i \cap [p_i, +\infty[$. Note that in all our results the set X_i is convex, and in that case single-peakedness simply means that $\succeq_i$ is strictly convex.

We write $\mathcal{SP}(X_i)$ for the set of such preferences, and the domain of preferences profiles as $\mathcal{SP}(X_N) = \Pi_{i \in N} \mathcal{SP}(X_i)$. A preference profile is $\succeq = (\succeq_i)_{i \in N} \in \mathcal{SP}(X_N)$ and $p = (p_i)_{i \in N} \in X_N$ is a profile of individual peaks.

Definition 1 *A one-dimensional allocation problem is a triple* $(N, X, \succeq)$ *where* X *is closed and* $\succeq \in \mathcal{SP}(X_N)$.

Definition 2 *Fixing the pair* (N, X)*, a revelation mechanism (aka a rule) is a (single-valued) mapping* F *choosing a feasible allocation for each allocation problem*

$$F : \mathcal{SP}(X_N) \to X \text{ written as } F(\succeq) = x$$

A revelation mechanism F is ***peak-only*** *if it is described by a (single-valued) mapping*

$$f : X_N \rightarrow X \text{ written as } f(p) = x$$

such that for all $\succeq \in \mathcal{SP}(X_N)$ with profile of peaks $p \in X_N$ we have $F(\succeq) = f(p)$.

A *peak-only* revelation mechanism is a particularly simple direct revelation mechanism because participants need to report only their peak, so an agent does not even need to figure out how she compares allocations across her peak to participate.

We will apply our main result to the following examples and discuss the corresponding good mechanisms in Sections 8,9. Start with three examples already in the literature.

Example 1 *voting* Here X is a closed interval of the diagonal $\Delta = \{x \in \mathbb{R}^N | x_i = x_j \text{ for all } i, j \in N\}$.

Example 2 *non disposable division* The feasible set is the simplex $X = \{x \in \mathbb{R}^N | x \geq 0 \text{ and } \sum_{i \in N} x_i = 1\}$.

Example 3 *supply-demand* This is the problem, closely related to the above, where each agent i can be a supplier or a demander of the non disposable commodity. For simplicity we normalize initial endowment at zero and ignore bankruptcy constraints, so that $X = \{x \in \mathbb{R}^N | \sum_{i \in N} x_i = 0\}$. If $p_i < 0$ (resp. $p_i > 0$) agent i wishes to be a net supplier (resp. demander) of the commodity.

The next two examples are new.

Example 4 *relocation* Initially the agents live at 0; they wish to relocate somewhere on the real line. The stand alone cost of moving agent i to location x_i is x_i^2, and in addition there are externalities, positive or negative, to locate x_i near x_j. The agents share a total relocation budget of 1. Formally

$$x \in X \overset{def}{\Longleftrightarrow} \sum_{i \in N} x_i^2 + \pi \sum_{i,j \in N} (x_i - x_j)^2 \leq 1$$

The externality factor π is positive if for instance some construction costs (pipes) of two near homes are shared; it is negative if the term $\pi(x_i - x_j)^2$ covers the cost of isolating homes i, j from one another.

Example 5 *bilateral workload*

We have a fixed partition of N as $L \cup R$, and we set $X = \{x \in \mathbb{R}^N | x \geq 0 \text{ and } \sum_{i \in L} x_i = \sum_{j \in R} x_j\}$. We think of two teams L, R who must coordinate their total work-load (as in a production chain where L is upstream of R), while teammates share the total load. If R consists of a single "manager" we have a moneyless version of the principal agent problem, where the principal wishes to adjust total output to his own target level, while the workers' individual targets should also be taken into account.

5 Efficiency and Incentives

The properties defined in this section are quite standard and we do not comment on them for brevity.

Definition 3 *The revelation mechanism F at (N, X) is*
Efficient (EFF) *if for any $\succeq \in \mathcal{SP}(X_N)$ the allocation $x = F(\succeq)$ is Pareto optimal at $\succeq$;*
Continuous (CONT) *if F is continuous for the topology of the Hausdorf distance on $\mathcal{SP}(X_N)$; if F is peak only this simply means that f is continuous in $\mathbb{R}^N$.*

Next we define three increasingly more demanding versions of incentive compatibility for revelation mechanisms. Fixing (N, X), a profile of preferences $\succeq \in \mathcal{SP}(X_N)$ and a coalition $M \subseteq N$, we say that M *can misreport* at $\succeq$ if there is some $\succeq'_{[M]} \overset{def}{=} (\succeq'_i)_{i \in M} \in \mathcal{SP}(X_M)$ such that $x'_i \succ_i x_i$ for all $i \in M$, where $x = F(\succeq)$ and $x' = F(\succeq'_{[M]}, \succeq_{[N \smallsetminus M]})$. We say that M *can weakly misreport* at $\succeq$ if under the same premises we have $x'_i \succeq_i x_i$ for all $i \in M$ with at least one is a strict preference.

Definition 4 *The revelation mechanism F is*
Strategyproof (SP) *if no single agent can misreport at any profile in $\mathcal{SP}(X_N)$;*
Groupstrategyproof (GSP) *if no coalition can misreport at any profile in $\mathcal{SP}(X_N)$;*
Strongly Groupstrategyproof (SGSP) *if no coalition can weakly misreport at any profile in $\mathcal{SP}(X_N)$.*

In general GSP (or SGSP) is considerably stronger than SP, the voting problem being an exception.[3] We recall two well known facts useful below.

Lemma 1 *We fix (N, X) and a revelation mechanism F at (N, X) (Definition 2)*
i) If F is strategyproof and continuous, then it is peak-only.

[3] See [3] for a detailed discussion of the connections between the two concepts in domains more general than singlepeaked.

ii) If F is peak-only, the mapping $p \to f(p)$ representing F is weakly increasing and "uncompromising": for all $p \in X_N$ and all $i \in N$

$$f_i(p) = x_i < p_i \text{ (resp. } x_i > p_i) \Longrightarrow$$

$$f_i(p'_i, p_{-i}) = x_i \text{ for all } p'_i \geq x_i \text{ (resp. } p'_i \leq x_i)$$

Proof: For statement $i)$ we fix $i \in N$ and $\succeq_{[N \setminus i]} \in \mathcal{SP}(X_{N \setminus i})$. Assume $\succeq_i^1, \succeq_i^2 \in \mathcal{SP}(X_i)$ have the same peak p_i but $x_i^1 = F_i(\succeq_i^1, \succeq_{N \setminus i}) \neq x_i^2 = F_i(\succeq_i^2, \succeq_{N \setminus i})$ then derive a contradiction. By SP the peak p_i must be strictly between x_i^1 and x_i^2, else agent i can misreport at one of $(\succeq_i^1, \succeq_{[N \setminus i]})$ or $(\succeq_i^2, \succeq_{[N \setminus i]})$. But CONT implies that the range of $\succeq_i \to F_i(\succeq_i, \succeq_{[N \setminus i]})$ is connected so it contains p_i and this yields a profitable misreport at both $(\succeq_i^1, \succeq_{[N \setminus i]})$ and $(\succeq_i^2, \succeq_{[N \setminus i]})$. The standard proof of the statement $ii)$ is omitted for brevity.■

It is a folk result that a *fixed priority* mechanism (also called *serial dictatorship*) is both efficient and groupstrategyproof. The simplest example is a dictatorial voting rule. In our general model define the *slice* of X at $\widetilde{x}_{[M]}$ as $X[\widetilde{x}_{[M]}] = \{x_{[N \setminus M]} \in \mathbb{R}^{N \setminus M} | (\widetilde{x}_{[M]}, x_{[N \setminus M]}) \in X\}$: it is closed and possibly empty. Given the priority ordering $1, 2, \cdots$, the mechanism gives her peak p_1 to agent 1 (this is feasible by definition of X_1) then to agent 2 his best allocation x_2 in (the projection on the 2d coordinate of) $X[p_1]$; next to agent 3 her best allocation x_3 in (the projection on the 3rd coordinate of) $X[(p_1, x_2)]$; and so on. *If X is convex*, each step is well defined as we maximize a single-peaked preference in a closed real interval. Then the mechanism is peak-only, efficient and strongly groupstrategyproof (instead of just GSP). It is continuous as well, but to prove it requires arguments similar to those of steps 6 and 9 in the proof of the main theorem.[4]

The strength of our Theorem is to achieve all the properties in Definitions 3,4 in a mechanism treating the participants fairly.

6 Fairness

We adapt the familiar "anonymity" property (aka horizontal equity) to our context where the set X itself may not treat all agents symmetrically. This requires a few definitions. Let $S(N)$ be the set of all permutations σ of N. Permuting coordinates according to σ changes x to $x^\sigma = (x_{\sigma(i)})_{i \in N}$ and $\succeq$ to $\succeq^\sigma = (\succeq_{\sigma(i)})_{i \in N}$. We call $\sigma \in S(N)$ a *symmetry of X* if $X^\sigma = X$, and

[4] It is of course possible to define the mechanism when X is not convex, and it retains the properties EFF and GSP, but is not necessarily SGSP, peak-only, or continuous.

write their set $S(N, X)$. We call ω a symmetric element of X if $\omega \in X$ and $\omega^\sigma = \omega$ for all $\sigma \in S(N, X)$.

In Examples 1 to 4 we have $S(N, X) = S(N)$ and we speak of a fully symmetric set X; in Example 5 $S(N, Z)$ contains the permutations leaving both L and R unchanged, but not those swapping agents between the two groups.

Of special interest are the simple permutations τ_{ij} exchanging i and j while leaving all other coordinates constant. If τ_{ij} is a symmetry of X we think of agents i and j as having identical opportunities in X so then the No Envy test where i compare his allocation to j's allocation is meaningful.

Our three fairness requirements are not logically connected to one another.

Definition 5 *The revelation mechanism* F at (N, X) *is*
Symmetric (SYM) *if for every* $\sigma \in S(N, X)$ *we have* $\{F(\succeq) = x \implies F(\succeq^\sigma) = x^\sigma\}$;
Envy-Free (EF) *if whenever* $\tau_{ij} \in S(N, X)$ *and* $F(\succeq) = x$ *we have* $x_i \succeq_i x_j$;

Given an allocation $\omega \in X$ *the mechanism* F
ω**-Guaranteed (ω-G)** *if* $F(\succeq) = x$ *implies* $x_i \succeq_i \omega_i$ *for all* i.

Just like in axiomatic bargaining, the ω-G property makes sense when ω is a default option (e.g., status quo ante) that each agent can revert to.

7 Main result and the uniform-gains rules

Theorem *Fix* (N, X) *and a symmetric allocation* $\omega \in X$*. If* X *is closed and convex in* $\mathbb{R}^N$ *there exists at least one peak-only mechanism* f^ω *at* (N, X) *that is Efficient, Symmetric, Envy-Free, Continuous, SGSP and* ω*-Guaranteed.*

To define the canonical *uniform-gains rule* proving the result we introduce some notation. Write $\succeq_{lexic}$ for the lexicographic ordering of $\mathbb{R}^n$, maximizing first coordinate 1, then coordinate 2 conditional on the previous maximization, and so on. The *leximin* ordering $\succeq_{lxmin}$ of $\mathbb{R}^N$ is a symmetric version of $\succeq_{lexic}$. For any $x, y \in \mathbb{R}^N$

$$x \succeq_{lxmin} y \overset{def}{\Longleftrightarrow} x^* \succeq_{lexic} y^* \quad (1)$$

where $x^* \in \mathbb{R}^n$ has the same set of coordinates as x (including possible repetitions) rearranged increasingly, and is written as follows: $\min_N x_i = x^{*1} \leq x^{*2} \leq \cdots \leq x^{*n} = \max_N x_i$.

Clearly $\succeq_{lxmin}$ is an ordering (complete, transitive) of $\mathbb{R}^N$, but is discontinuous and cannot be represented by a utility function. Over a compact set its maximum exists but may not be unique, however its maximum over a *convex* compact set is unique.[5]

We pick an arbitrary ω in X, not necessarily symmetric, and define the peak-only mechanism f^ω, meeting all properties in the Theorem except perhaps SYM and EF. It is then easy to check SYM when ω is symmetric in X, and EF when two agents are interchangeable in X.

In $\mathbb{R}^N$ we use the notation $[a,b] \stackrel{def}{=} \{x| \min\{a_i,b_i\} \leq x_i \leq \max\{a_i,b_i\}$ for all $i\}$ and $|a| = (|a_i|)_{i\in N}$. Given a profile of peaks p the rule f^ω chooses an allocation x in $[\omega,p]$. The vector $|x-\omega|$ is the profile of gains from the benchmark ω, using the distance $|x_i - \omega_i|$ as an arbitrary cardinalization of these ordinal welfare gains. We equalize gains across agents as much as permitted by feasibility:

$$f^\omega(p) = x \stackrel{def}{\Longleftrightarrow} \{x \in X \cap [\omega,p] \text{ and } |x-\omega| = \arg \max_{\Delta(\omega,p)} \succeq_{lxmin}\} \qquad (2)$$

where

$$z \in \Delta(\omega,p) \stackrel{def}{\Longleftrightarrow} \{z = |x-\omega| \text{ for some } x \in X \cap [\omega,p]\}$$

The allocation $f^\omega(p)$ is well defined because $\Delta(\omega,p)$ is convex and compact, so the maximum of $\succeq_{lxmin}$ exists and is unique. We show in Section 11 that f^ω meets EFF, CONT and SGSP. Continuity turns out to be the hardest part of the proof.

The convexity of X is a sufficient condition for the existence of a good mechanism (meeting EFF, SYM, EF, CONT and SGSP), but it is by no means a necessary condition. We give in the next section a two person example of a good mechanism when X is a non convex subset of $\mathbb{R}^2$: see Remark 1 in Subsection 8.3.

On the other hand for some non convex sets X even Efficiency, Strategyproofness, and Continuity are incompatible. Figure 1 explains this in a two person example. If such a mechanism exists it is peak-only by Lemma 1. Say the profile of peaks is p and agent 1 reports c_1 instead of p_1, while agent 2 reports p_2: by EFF we have $f(c) = c$ so agent 1 can achieve c_1, as well as d_1 by a similar argument. Set $f_1(p) = x_1$ and note that $x_1 > p_1$ would contradict SP because there is a preference with peak p_1 ranking c_1 above x_1; and $x_1 < p_1$ is similarly impossible, so we conclude $f_1(p) = p_1$. The same argument for agent 2 gives $f_2(p) = p_2$ and we reach a contradiction.

[5] We recall the known argument (Lemma 1.1 in [32]) in step 1 of the proof, Section 10.

We turn to the family of fully symmetric problems, where we can describe the set of good mechanisms in some details.

8 Fully symmetric problems

When all permutations of N are symmetries of X, $S(N, X) = S(N)$, we say that (N, X) is a *fully symmetric* problem. All agents have the same feasible set X_i and Envy-Freeness applies to every pair of agents.

The affine space $H[X]$ spanned by X is also symmetric in all coordinates, and if X is not a singleton there are only three possibilities: $H[X]$ could be the (one-dimensional) diagonal D of $\mathbb{R}^N$; it could be a $(n-1)$-dimensional subspace orthogonal to D; or it could have full dimension: $H[X] = \mathbb{R}^N$.(We omit the straightforward proof of this statement).

In the first case X is a closed interval of D and we have a *voting problem*. In the second case the sum $\sum_N x_i$ is constant in X and we speak of a *generalized division problem*. The case $H[X] = \mathbb{R}^N$ yields an entirely new class of problems.

8.1 Voting

Let X_0 be the set of individual allocations common to all agents: a fule f can be simply written as a mapping from X_0^N into X_0. Any allocation $\omega \in X \subseteq D$ is symmetric: $\omega_i = \omega_0 \in X_0$ for all i. To read definition (2) fix a profile of peaks $p \in X_0^N$ and some $x \in X \cap [\omega, p]$ so that $x_i = x_0$ for all i. If there are agents i, j such that $p_i \leq \omega_0 \leq p_j$ then $x = \omega$ because $x \in [\omega, p]$ implies $p_i \leq x_i \leq \omega_0 \leq x_j \leq p_j$. If $\omega_0 \leq p_i$ for all i then $\omega_0 \leq x_0 \leq p^{*1}$ and $x_0 - \omega_0$ is maximal at $f^\omega(p) = p^{*1}$; similarly if $p_i \leq \omega_0$ for all i we have $f^\omega(p) = p^{*n}$. We just proved

Proposition 1 *Given (N, X_0) and $\omega_0 \in X_0$ the rule f^ω defined by (2) is*

$$f^\omega(p) = \text{median}\{p^{*1}, p^{*n}, \omega_0\}$$

We already know that a voting rule in (N, X_0) is Efficient, Symmetric, and Strategyproof if and only if it is a *generalized median* rule ([31], [43]). Such a rule is defined by the choice of $(n-1)$ arbitrary parameters α_k in $X_0, 1 \leq k \leq n-1$, interpreted as *fixed ballots*[6] and it picks the median of the fixed and the live ballots:

$$f(p) = \text{median}\{p_i, i \in N; \alpha_k, 1 \leq k \leq n-1\}$$

[6] The parameter α_k could be $\pm\infty$ if this is an endpoint of X_0.

(they also meet SGSP and CONT).

We see that f^ω is such a rule when all $n-1$ fixed ballots α_k are the status quo ω_0.

8.2 Dividing

Now that $H[X]$ is orthogonal to the diagonal D of $\mathbb{R}^N$, so that X takes the form $X = \{\sum_N x_i = \beta\} \cap C$ where β is a real number and C is convex, closed and fully symmetric and not diagonal. There is only one symmetric point ω in X, i.e., equal split: $\omega_i = \frac{1}{n}\beta$ for each i.

Proposition 2: *Given* (N, X) *where* $X = \{\sum_N x_i = \beta\} \cap C$ *is a fully symmetric division problem, there is a unique rule that is Efficient, Symmetric, Continuous and SGSP: it is* f^ω *with* $\omega_i = \frac{1}{n}\beta$ *for all* i.

In Example 2 X is the simplex $X^s(\beta) = \{x \geq 0, \sum_N x_i = \beta\}$ and earlier results show there is a single mechanism Efficient, Symmetric and SP: the *uniform rationing* rule ([42], [17]). By Proposition 2, our f^ω is precisely the same rule. Compare now the definition of f^ω in (2) with the standard definition of uniform rationing.

For the latter the key observation is that efficient allocations are *one-sided*. Fixing a profile of peaks p, if we have excess demand, $\sum_N p_i > \beta$, the allocation x is efficient if and only if $x_i \leq p_i$ for all i; if we have excess supply, $\sum_N p_i < \beta$, efficiency means $x_i \geq p_i$ for all i. Then the rationing rule h equalizes the shares x_i, conditional upon $x_i \leq p_i$ under excess demand, and upon $x_i \geq p_i$ under excess supply. Formally $h(p)$ is captured by a parameter $\lambda \in [0, \beta]$ such that

$$\begin{aligned} \text{if } \sum_N p_i &\geq \beta : h_i(p) = \min\{\lambda, p_i\} \text{ for all } i \\ \text{or if } \sum_N p_i &\leq \beta : h_i(p) = \max\{\lambda, p_i\} \text{ for all } i \end{aligned} \tag{3}$$

(The reader can check directly that $h = f^\omega$)

In the general problems covered by Proposition 2, efficient allocations may no longer be one-sided. For instance we are dividing 100 shares in some joint venture between four partners, and must make sure no pair of agents owns $\frac{2}{3}$ of the shares: the allocation $x \in \mathbb{R}^4_+$ is feasible iff

$$\sum_1^4 x_i = 100 \text{ and } x_i + x_j \leq 66 \text{ for all } i \neq j$$

Then at the profile of peaks $p = (10, 15, 35, 40)$ the allocation $x = (17, 17, 30, 36)$ is efficient.

This complicates the proof of Proposition 2 and accounts for the additional assumptions (SGSP and CONT) not necessary in the Sprumont-Ching characterization. Still a plausible conjecture is that Proposition 3 holds when SGSP is replaced by SP.

Consider finally the supply-demand Example 3. We have $X = \{\sum_N x_i = 0\}$, so $\omega = 0$ is the only symmetric allocation in X. Here f^0 rations uniformly the long side: if $\sum_N p_i > 0$ total demand $\sum_{i:p_i>0} p_i$ exceeds total supply $\sum_{i:p_i<0} |p_i|$ so each supplier unloads her peak amount while the demanders share the total supply according to the uniform rule above; and vice versa if supply exceeds demand. This rule is characterized in [28] (see also [11]) by Efficiency, Voluntary Trade (0-G) and SP: efficient allocations must be one-sided so that the proof in [17] can be adapted. Proposition 2 is an alternative characterization where Voluntary Trade is replaced by Symmetry plus Continuity, and SP by SGSP.

8.3 Full dimension problems

Proposition 3

i) If $n = 2$ and the closed, convex subset X of $\mathbb{R}^N$ is symmetric and of dimension 2, then a mechanism F (Definition 2) is Efficient, Symmetric, Continuous and SGSP if and only if it is f^ω for some symmetric allocation ω in X.

ii) If $n \geq 3$ and the closed, convex subset X of $\mathbb{R}^N$ is symmetric and of dimension n, then the set of mechanisms Efficient, Symmetric, Continuous and SGSP is of infinite dimension (while the symmetric rules f^ω form a subset of dimension 1).

The proof of statement *i*) is explained below in an instance of the relocation Example 4. Section 10 has the rest of the proof. We assume positive externalities when the two agents live close to each other so that

$$X = \{x_1^2 + x_2^2 - \frac{8}{5}x_1 x_2 \leq 1\} \tag{4}$$

Figure 2 represents the elliptic feasible set X where $X_i = [-\frac{5}{3}, \frac{5}{3}]$ for $i = 1, 2$. Also represented are the symmetric point $\omega = (\frac{1}{3}, \frac{1}{3})$ and the four boundary points a, b, c, d of X critical to the construction of f^ω. By EFF we only need to describe $f^\omega(p)$ when p is outside X. Suppose p is to the NorthEast (NE) of a. Outcome a is efficient at p and inside $[\omega, p]$; it also equalizes the benefits $|a_i - \omega_i|$ therefore $f^\omega(p) = a$. Similar arguments show that $f^\omega(p) = b$ for

p in the NW of b, $f^\omega(p) = c$ if p is SW of c and $f^\omega(p) = d$ if it is SE of d. Now take p SouthEast of ω but SW of d: at outcome x shown in Figure 2 the vector $(|x_1 - \omega_1|, |x_2 - \omega_2|) = (|p_1 - \omega_1|, |x_2 - \omega_2|)$ is leximin optimal for $x \in [\omega, p]$, thus $f^\omega(p) = x$. We see now that for any p outside X that is West of d, East of c and South of ω, agent 1 gets her peak allocation and, conditional on this, x_2 is best for agent 2. Similar arguments in the three other remaining regions complete the description of f^ω.

We show now that, conversely, any rule F meeting EFF, SYM, CONT and SGSP is precisely f^ω for some ω in the diagonal of X. The proof works by focusing on the choice of F at the four corners of X_{12} namely $A = (\frac{5}{3}, \frac{5}{3})$ in the NE corner, B in the NW, and so on. By Lemma 1 F is single-peaked so we write it f. By EFF and SYM we have $f(A) = a$, $f(C) = c$. Now by efficiency $f(B)$ is some point b on the NW frontier of X, and by symmetry $f(D) = d$ obtains from b by exchanging its coordinates. Call ω the intersection of the line bd and the diagonal: we claim that $f = f^\omega$.

Consider first the rectangle $[B, b]$: by uncompromisingness (Lemma 1 statement *ii*)) $f_1(p_1, B_2) = b_1$ for any $p_1 \in [B_1, b_1]$; by SGSP $f_2(p_1, B_2) = b_2$ as well, else $\{1, 2\}$ can weakly manipulate either at B or at (p_1, B_2). So $f(p) = b$ along the top edge of $[B, b]$. Repeating this argument we see that $f(p) = b$ still holds along its left edge, and then inside $[B, b]$ as well. Similarly $f = f^\omega$ in the three other rectangles $[A, a], [C, c]$ and $[D, d]$. Now consider the point p in Figure 2 that is neither in X nor in any of these four rectangles. By efficiency $f(p) = z$ is on the frontier of X between y and x. We assume $z_1 < x_1 = p_1$ and derive a contradiction. By uncompromisingness we get $f_1(\frac{5}{3}, p_2) = f_1(p) = z_1$ and by SGSP as above we have $f_2(\frac{5}{3}, p_2) = f_2(p)$ as well: but $(\frac{5}{3}, p_2) \in [D, d]$ so $f(\frac{5}{3}, p_2) = d$, contradiction. We conclude that f and f^ω coincide in the triangular region bordered by $[D, d]$ and the SE frontier of X. Finally we repeat this argument in the seven other relevant regions.

Remark 1 Figure 3 shows a non convex feasible set X where the same construction as above delivers the good mechanism f^ω (still defined by (2)). It goes to show that convexity is not a necessary condition for the existence of a good mechanism in the sense of the Theorem.

9 An embarrassment of riches

We consider the simplest non trivial instance of the bilateral workload Example 5 with two agents on one side and one on the other: $L = \{1, 2\}$ and $R = \{3\}$. Thus $X = \{x \in \mathbb{R}^3_+ | x_1 + x_2 = x_3\}$. We find that the set of good rules is quite rich and worthy of further research.

This makes a different point than statement $ii)$ in Proposition 3: in the proof of that result we construct a large set of good rules by drawing a wedge between agent i's allocations above the default ω_i, or below; these new rules are mere variants of the canonical uniform gains rule. Here we find instead a menu of genuinely different power-sharing scenarios between the three participants.

Let f be a good rule, namely meeting EFF, SGSP, SYM and CONT. For a prolile $p \in \mathbb{R}_+^3$ we write $f(p) = (x_1, x_2, t(p))$ where $x_3 = t(p)$ is the amount that agents $1, 2$ have to share. It is easy to check that they do so by the uniform rationing rule (by using the argument in Step 1 of the proof of Proposition 2), therefore the function $t(\cdot)$ determines f entirely. Efficiency amounts to $t(p) \in [p_1 + p_2, p_3]$, and Symmetry means that $t(p_1, p_2, p_3)$ is symmetric in p_1, p_2.Fixing p_1, p_2 the mapping $p_3 \to t(p)$ must ensure agent 3's truthfulness, which means that it is the projection of p_3 on an interval independent of p_3.

Putting these facts together we get the general form

$$t(p) = \text{median}\{p_3, J_-(p_1, p_2), J_+(p_1, p_2)\} \tag{5}$$

where $J_{-,+}$ are symmetric, continuous functions such that

$$0 \leq J_-(p_1, p_2) \leq p_1 + p_2 \leq J_+(p_1, p_2) \tag{6}$$

Of course SGSP imposes some further constraints on $J_{-,+}$.

We describe three families of rules where SGSP holds. A full description reveals a set of choices much larger but not necessarily more interesting.

First family of good rules

They all guarantee a benchmark allocation $\omega = (\alpha, \alpha, 2\alpha) \in X$. Think of a supply-demand model similar to Example 3 between demanders $1, 2$ and supplier 3 where ω is the profile of initial endowments. Then

$$t(p) = \text{median}\{p_1 + p_2, p_3, 2\alpha\} \tag{7}$$

corresponds to the rule giving its peak to the short side and rationing the long side (here $J_-(p_1, p_2) = \min\{p_1 + p_2, 2\alpha\}$ and $J_+(p_1, p_2) = \max\{p_1 + p_2, 2\alpha\}$). We let the reader check the ω-G property.

The canonical rule f^ω also guarantees ω, but proves to be more complicated than the "rationing" rule (7). Straightforward computations from definition (2) give the following J_-, J_+ in (5):

$$J_-(p_1, p_2) = p_1 + p_2 \text{ if } 2p_1 + p_2, p_1 + 2p_2 \leq 3\alpha$$

$$= \alpha + \frac{1}{2}\min\{p_1, \alpha\} + \frac{1}{2}\min\{p_2, \alpha\} \text{ otherwise}$$

and

$$J_+(p_1, p_2) = p_1 + p_2 \text{ if } 2p_1 + p_2, p_1 + 2p_2 \geq 3\alpha$$
$$= \alpha + \frac{1}{2}\max\{p_1, \alpha\} + \frac{1}{2}\max\{p_2, \alpha\} \text{ otherwise}$$

This rule coincides with the rule (7) if $p_1, p_2 \leq \alpha$ and if $\alpha \leq p_1, p_2$. But if for instance $p_3 < 2\alpha < p_1 + p_2$ and $p_1 < \alpha < p_2$, then $t(p)$ is smaller here method than under rule (7) which may or may not favor agent 3 or agent 1.

Second family of good rules

We now run a vote between the three agents to determine $t(p)$: thus agent $i = 1, 2$ reports $2p_i$, because $t(p) = 2p_i$ guarantees $x_i = p_i$. The simplest rule is majority voting

$$t(p) = \text{median}\{2p_1, 2p_2, p_3\} = \text{median}\{2p^{*1}, 2p^{*2}, p_3\} \tag{8}$$

In the family of strategyproof voting rules $p \to t(p)$ respecting the symmetry between 1 and 2, the ones ensuring efficiency (6) take the form

$$t(p) = \text{median}\{\min\{2p^{*1}, \alpha\}, \max\{2p^{*2}, \beta\}, p_3\}$$

for some constants α, β such that $\alpha \leq \beta$. Note that agent 3 can enforce any x_3 in $[\alpha, \beta]$ while agents $1, 2$ together can only force $t(p)$ below β or above α.[7]

Third family of good rules

We fix $\gamma, \delta \geq 0$ and apply the general formula (5) with the following functions:

$$J_-(p_1, p_2) = \min\{p_1, (p_2 + \gamma)\} + \min\{(p_1 + \gamma), p_2\}$$
$$J_+(p_1, p_2) = \max\{p_1, (p_2 - \delta)\} + \max\{(p_1 - \delta), p_2\}$$

For $\gamma = \delta = 0$ this is the simple majority rule (8). For general parameters γ, δ the rule gives full power to agents $1, 2$ if their peaks are not too different: $t(p) = p_1 + p_2$ if $|p_1 - p_2| \leq \min\{\gamma, \delta\}$; if, for instance, $p_1 \geq p_2 + \max\{\gamma, \delta\}$ then $t(p) = \text{median}\{2p_1 - \delta, 2p_2 + \gamma, p_3\}$.

[7] A variant is the rule $t(p) =$median$\{\min\{p_1 + p_2, 2\alpha\}, \max\{p_1 + p_2, 2\beta\}, p_3\}$ where agent 3 can also force x_3 anywhere in $[2\alpha, 2\beta]$, while if agent $i = 1, 2$ reports $p_i \in [\alpha, \beta]$ she guarantees only that x_i is somewhere in $[\alpha, \beta]$.

Conversely if $\beta \leq \alpha$ then $t(p) = p_1 + p_2$ if $p_1 + p_2 \in [2\alpha, 2\beta]$, while the report $p_3 \in [2\alpha, 2\beta]$ only guarantees $x_3 \in [2\alpha, 2\beta]$.

10 Proofs

10.1 Main Theorem

Step 1 *The leximin ordering*

Recall from section 6 the notation $\mathbb{R}^N \ni x \to x^* \in \mathbb{R}^n$ where x^* simply rearranges the coordinates of x increasingly. The *leximin* ordering $\succeq_{lxmin}$ of $\mathbb{R}^N$ applies $\succeq_{lexic}$to x^* as stated in equation (1). It is a *separable* ordering, which means that for any $x, y \in \mathbb{R}^N$ and any $i \in N$

$$\{x \succeq_{lxmin} y \text{ and } x_i = y_i\} \Longrightarrow x_{-i} \succeq_{lxmin} y_{-i}$$

(where the second inequality is in $\mathbb{R}^{N \diagdown i}$). Check now that $\succeq_{lxmin}$ has a unique maximum over any *convex* and compact set C of $\mathbb{R}^N$. Suppose instead that x and y are two such maximizers so that $x^{*1} = y^{*1} = a$. Compare $S = \{i \in N | x_i = a\}$ with $T = \{j \in N | y_j = a\}$. If they are disjoint we have for all $k \in N$ $a \leq \min\{x_k, y_k\} < \max\{x_k, y_k\}$ implying $\min_{k \in N}(\frac{x+y}{2})_k > a$ and contradicting the optimality of x. Thus there is an agent labeled 1 in $S \cap T$ and such that $x_1 = y_1 = a$. Then by separabilty, x_{-1} and y_{-1} maximize $\succeq_{lxmin}$ in the slice $C[a_{[1]}]$ and we can proceed by induction on $|N|$.

Here is another fact useful below with a similar proof (omitted). For all $u, v \in \mathbb{R}^N$

$$u \succeq_{lxmin} v \Longrightarrow (\lambda u + (1-\lambda)v) \succeq_{lxmin} v \text{ for all } \lambda, 0 \leq \lambda \leq 1 \qquad (9)$$

Throughout the rest of the proof we fix (N, X) with X convex and closed.

Step 2 *Efficient allocations*

Let $\mathcal{T}$ be the set of triples $\tau = (S_0, S_+, S_-)$ of pairwise disjoint subsets of N covering N.where up to two components of τ can be empty (if all three are non empty τ is a partition of N). The signature $\tau = s(y)$ of $y \in \mathbb{R}^N$ is given by $S_0 = \{i \in N | y_i = 0\}$, $S_+ = \{i \in N | y_i > 0\}$, $S_- = \{i \in N | y_i < 0\}$. We define a transitive but incomplete ordering $\trianglerighteq$ on $\mathcal{T}$ by

$$\tau^1 \trianglerighteq \tau^2 \overset{def}{\Longleftrightarrow} \{S_0^2 \supseteq S_0^1 \text{ , } S_+^2 \subseteq S_+^1, \, S_-^2 \subseteq S_-^1\}$$

and $\triangleright$ is the strict component of $\trianglerighteq$.

Fixing $\tau \in \mathcal{T}$ we define the τ-boundary of X as follows

$$\partial^\tau(X) = \{x \in X | \text{ for all } y \; \{y \neq x \text{ and } s(y-x) \trianglerighteq \tau\} \Longrightarrow y \notin X\}$$

Lemma 2 *Fix $p \in X_N$. If $p \in X$ then $x = p$ is the only Pareto optimal allocation. If $p \notin X$ then $x \in X$ is Pareto optimal for every profile $\succeq \in \Pi_{i \in N} \mathcal{SP}(X_i)$ with peaks p* **if and only if** $x \in \partial^{s(p-x)}(X)$.

Proof. The first statement is clear. Next assume $p \notin X$ and pick $x \in X$ such that $x \notin \partial^{s(p-x)}(X)$. Then there exists $y \in X \setminus x$ such that $s(y-x) \trianglerighteq s(p-x)$. This implies $y_i = x_i$ for each i such that $x_i = p_i$, and for all j

$$y_j > x_j \Longrightarrow p_j > x_j \text{ and } y_j < x_j \Longrightarrow p_j < x_j$$

From $y \neq x$ we see that not both S_+ and S_- are empty at $y-x$, therefore for $\varepsilon > 0$ small enough $\varepsilon y + (1-\varepsilon)x$ stays in X and is a Pareto improvement of x.

Conversely if $x \in X$ is Pareto inferior to $y \in X$ for every relevant profile $\succeq$ we get $x_i = p_i \Longrightarrow y_i = x_i$, and $y_j > x_j \Longrightarrow p_j \geq y_j \Longrightarrow p_j > x_j$, and similarly $y_j < x_j \Longrightarrow p_j < x_j$, so we conclude $x \notin \partial^{s(p-x)}(X)$.

Step 3 *Defining* f^ω

For $a \in \mathbb{R}^N$ we write $|a| = (|a_i|)_{i \in N}$ and for any a, b we define the rectangle $[a,b] = \{x \in \mathbb{R}^N | \min\{a_i, b_i\} \leq x_i \leq \max\{a_i, b_i\} \text{ for all } i\}$.

We fix a point $\omega \in X$. Then for all $p \in \mathbb{R}^N$ we define

$$f^\omega(p) = x \overset{def}{\Longleftrightarrow} \{\ x \in X \cap [\omega, p] \text{ and } |x - \omega| = \arg \max_{\Delta(\omega,p)} \succeq_{lxmin}\}$$

where

$$y \in \Delta(\omega, p) \overset{def}{\Longleftrightarrow} y = |z - \omega| \text{ for some } z \in X \cap [\omega, p]$$

This is well defined because for any $x \in [\omega, p]$ we have $s(x-\omega) \trianglerighteq s(p-\omega)$ therefore in $[\omega, p]$ each $|x_i - \omega_i|$ is either $x_i - \omega_i$ or $\omega_i - x_i$, so the mapping $x \to |x - \omega|$ is linear and invertible in $X \cap [\omega, p]$ and its image $\Delta(\omega, p)$ is convex and compact. By Step 1 $\succeq_{lxmin}$ has a unique maximum y in $\Delta(\omega, p)$, which comes from a unique x in $X \cap [\omega, p]$.

Step 4 f^ω *is efficient*

Fix p and set $x = f^\omega(p)$. If $p \in X$ then the maximum of $\succeq_{lxmin}$ on $\Delta(\omega, p)$ is clearly $|p - \omega|$ therefore $x = p$ as desired. Assume next $p \notin X$: by Lemma 2 we must check $x \in \partial^{s(p-x)}(X)$. Assume to the contrary there exists $y \in X \setminus x$ such that $s(y-x) \trianglerighteq s(p-x)$. Then $y_i = p_i$ whenever $x_i = p_i$, and if $y_i > x_i$ (resp. $y_i < x_i$) then $p_i > x_i$ (resp $p_i < x_i$). We see that for ε small enough $y' = (1-\varepsilon)x + \varepsilon y$ stays in $X \cap [\omega, p]$. For all i we have $|y'_i - \omega_i| = |y'_i - x_i| + |x_i - \omega_i| \geq |x_i - \omega_i|$, with a strict inequality if $y_i \neq x_i$ (which does happen). We conclude $|y' - \omega| \succ_{lxmin} |x - \omega|$ which is a contradiction.

Step 5 f^ω *is SGSP*

We fix ω and show first that f^{ω} meets a coalitional form of uncompromisingness (Lemma 1). For any $p, p' \in X_N$ with $x = f^{\omega}(p)$ we have

$$p' \in [x, p] \Longrightarrow f^{\omega}(p') = x \tag{10}$$

Together $x \in [\omega, p]$ and $p' \in [x, p]$ imply $x \in [\omega, p']$. Now $|x - \omega|$ maximizes (uniquely) $\succeq_{lxmin}$ over $\Delta(\omega, p)$, and is in $\Delta(\omega, p') \subseteq \Delta(\omega, p)$: hence $|x - \omega|$ maximizes $\succeq_{lxmin}$ over $\Delta(\omega, p')$, as was to be proved.

Next we fix $p \in X_N$ with $x = f^{\omega}(p)$, and consider a coalition $M \subseteq N$ changing all its reports to $p'_{[M]}$ (so $p'_i \neq p_i$ for all $i \in M$), and such that everyone in M weakly prefers $x' = f^{\omega}(p'_{[M]}, p_{[N \setminus M]})$ to x. We claim that this implies $x' = x$. Hence M, as well as any coalition larger than M, cannot weakly misreport at p and we are done.

To prove the claim, consider first an agent i such that $p_i = \omega_i$. By definition of f^{ω} we have $x_i = p_i$ hence $x'_i = x_i$ as well because agent i's welfare does not decrease. So at profile $(p'_{[M]}, p_{[N \setminus M]})$ agent i allocation is $x_i \neq p'_i$ and uncompromisingness (10) implies that everyone's allocation is unchanged if i reports instead $x_i = p_i$: $f^{\omega}(p'_{[M]}, p_{[N \setminus M]}) = f^{\omega}(p'_{[M \setminus i]}, p_{[(N \setminus M) \cup i]})$. Therefore we need only to prove the claim when $p_i \neq \omega_i$ for all i.

For easier reading we assume, without loss of generality, $p_i > \omega_i$ for all i, so that $\omega_i \leq x_i \leq p_i$ for all i. We must have $p'_i \geq x_i$ for all $i \in M$, as $p'_i < x_i$ implies $x'_i < x_i$ and agent i is strictly worse off at x'. We partition M as $M_+ \cup M_-$ where $p'_i > p_i \geq x_i$ in M_+, while $p_i > p'_i \geq x_i$ in M_- (one set $M_{+,-}$ could be empty).

The coordinate-wise minimum of p and $(p'_{[M]}, p_{[N \setminus M]})$ is $q = (p_{[M_+]}, p'_{[M_-]}, p_{[N \setminus M]})$. From $q \in [x, p]$ and (10) we get $x = f^{\omega}(q)$. To conclude the proof we assume $x' \neq x$ and derive a contradiction. From $\Delta(\omega, q) \subseteq \Delta(\omega, (p'_{[M]}, p_{[N \setminus M]}))$ and the definition of f^{ω} we get $(x' - \omega) \succ_{lxmin} (x - \omega)$. Check that for ε positive and some small enough the profile $\varepsilon x' + (1 - \varepsilon) x$ is in $\Delta(\omega, q)$. Indeed for all $i \notin M_+$ we have $\omega_i \leq x_i, x'_i \leq q_i$ by definition of q; for $i \in M_+$ such that $x_i < p_i = q_i$ we have $x_i \leq x'_i$ (because i weakly prefers x' to x) so the inequalities $\omega_i \leq \varepsilon x'_i + (1 - \varepsilon) x_i \leq q_i$ hold for ε small enough; and for $i \in M_+$ such that $x_i = p_i = q_i$ we have $x'_i = p_i$ (again because i weakly improves from x to x') so that $\varepsilon x'_i + (1 - \varepsilon) x_i = x_i$.

Applying finally property (9) to $u = x' - \omega$, $v = x - \omega$, and $\lambda = \varepsilon$, we get $((\varepsilon x' + (1 - \varepsilon) x) - \omega) \succeq_{lxmin} (x - \omega)$, contradicting $x = f^{\omega}(q)$ because $\varepsilon x' + (1 - \varepsilon) x \neq x$.

Step 6 f^{ω} *is continuous*

Define an orthant Θ of $\mathbb{R}^N$ by fixing the sign of each coordinate: Θ is described by n inequalities $x_i \leq 0$ or $x_i \geq 0$, one for each coordinate i.

It is enough to show that f^ω is continuous when $p-\omega$ varies in such an orthant, because the orthants are 2^n closed sets covering $\mathbb{R}^N$. Without loss of generality we focus on the orthant $\Theta = \mathbb{R}^N_+$, i.e., we prove continuity for the set of profiles p such that $p \geq \omega$. Here $f^\omega(p) - \omega$ maximizes $\succeq_{lxmin}$ over $(X-\omega) \cap [0, p-\omega])$. Using the normalisation $\omega = 0$, we are left with

$$f^\omega(p) = \arg \max_{X \cap [0,p]} \succeq_{lxmin}$$

We will apply repeatedly a simple version of Berge's maximum theorem. Let a, b vary in two metric spaces A, B; fix a real-valued function $a \to g(a)$ and a compact-valued function $b \to \Gamma(b)$ from B into A. If g is continuous and Γ is hemicontinuous (meaning both upper and lower hemicontinuous), then the real-valued function $\gamma(b) = \max\{g(a) | a \in \Gamma(b)\}$ is continuous as well.

For any $(q,p) \in (\mathbb{R}^N_+)^2$ we set $\Phi(q,p) = X \cap [q,p]$ and we claim that the convex-compact-valued function $(q,p) \to \Phi(q,p)$ is hemicontinuous on the closed convex subset of $(\mathbb{R}^N_+)^2$ where it is non empty. The proof of this claim is postponed to step 9 below.

We prove now that the mapping $p \to f^*(p)$ is continuous. Observe that $x \to x^*$ is continuous, then check that the first coordinate of f^*

$$f^{*1}(p) = \max\{x^{*1} | x \in \Phi(0,p)\}$$

is continuous: Berge's theorem applies because $x \to x^{*1}$ is continuous and $\Phi(0,p)$ is hemicontinuous. We use now the notation e^S for the vector $(e^S)_i = 1$ if $i \in S$ and 0 if not, to write f^{*2} as

$$f^{*2}(p) = \max\{x^{*2} | x \in \Phi(f^{*1}(p)e^N, p)\}$$

It is continuous by Berge's theorem because $x \to x^{*2}$ is continuous and $\Phi(f^{*1}(p)e^N, p)$ is hemicontinuous. Next we write

$$f^{*3}(p) = \max\left\{x^{*3} | x \in \cup_{i \in N} \Phi(f^{*1}(p)e^i + f^{*2}(p)e^{N \diagdown i}, p)\right\}$$

Here $\Phi(f^{*1}(p)e^i + f^{*2}(p)e^{N \diagdown i}, p)$ is hemicontinuous and hemicontinuity is preserved by union, so the same argument applies.

Next we define similarly $f^{*4}(p)$ in terms of the sets $\Phi(f^{*1}(p)e^i + f^{*2}(p)e^j + e^{N \diagdown \{i,j\}}, p)$ and so on. We omits the details.

Thus f^* is continuous and we show now that f is too. Fix $p \in \mathbb{R}^N_+$ and let $p^t, t = 1, 2, \cdots$, be a sequence converging to p: if w is a limit point of the sequence $f(p^t)$ (i.e., the limit of one of its subsequences) then $w \in \Phi(0,p)$

because the graph of Φ is closed. Moreover $f^*(p^t)$ converges to w^*, and to $f^*(p)$, by continuity of $x \to x^*$ and of f^*, respectively. Thus $w^* = f^*(p)$ hence w maximizes $\succeq_{lxmin}$ in $\Phi(0,p)$ and by Step 1 this unique maximum is $f(p)$.

Step 7 *f^ω is symmetric if ω is symmetric in X*

A symmetric point always exists: the set $S(N;X)$ of all symmetries of X is a group for the composition of permutations. Starting from an arbitrary element x of X, we set $\omega = \frac{1}{|S(N;X)|}\sum_{\sigma \in S(N;X)} x^\sigma$, which is in X because it is convex, and is clearly symmetric in X.

We check that f^ω is symmetric if (and only if) ω is symmetric. For any profile $p \in X_N$ we must show $f^\omega(p^\sigma) = f^\omega(p)^\sigma$ whenever $\sigma \in S(N;X)$. As $\succeq_{lxmin}$ is a symmetric ordering we have $\arg\max_{B^\sigma} \succeq_{lxmin} = (\arg\max_B \succeq_{lxmin})^\sigma$ for any set B where the maximum is unique, moreover if $x^\sigma = x$ then $\Delta(\omega, p^\sigma) = \Delta(\omega, p)^\sigma$.

Step 8 *f^ω is Envy-Free*

Assume $\tau_{ij} \in S(N,X)$. The desired property $x_i \succeq_i x_j$ is clear if p_i and p_j are on both sides of $\omega_i = \omega_j$ because for agent i allocation x_i is on the "good" side of ω_i while x_j is on the "bad" side. Now assume p_i and p_j are on the same side of ω_i, say $p_i, p_j \geq \omega_i$, and agent i envies x_j: then $p_i > x_i \geq \omega_i$ and $x_j > x_i$. Note that x_j may be larger or smaller than p_i. We consider now several allocations where coordinates other than i,j stay as in x, and for brevity we only mention these two coordinates: e.g., x is simply (x_i, x_j). By the symmetry assumption, $x' = (x_j, x_i)$ is in X and by convexity so is $x'' = (\lambda x_i + (1-\lambda)x_j, (1-\lambda)x_i + \lambda x_j)$. For λ small enough (in particular below$\frac{1}{2}$) the allocation $(|x''_i - \omega_i|, |x''_j - \omega_j|)$ is in $\Delta(\omega, p)$ (recall $x_i < p_i$) and the shift from $(|x_i - \omega_i|, |x_j - \omega_j|)$ to $(|x''_i - \omega_i|, |x''_j - \omega_j|)$ is a Pigou Dalton transfer hence it improves the leximin ordering.

Step 9 *hemicontinuity of* $(q,p) \to \Phi(q,p) = [q,p] \cap X$

Upper hemicontinuity is clear because the graph of Φ is closed. For lower hemicontinuity we use an auxiliary result. Consider a polyhedral-valued function $b \to H(b) = \{x \in \mathbb{R}^{m_2} | Ax \leq b\}$ where $b \in \mathbb{R}^{m_1}$ and A is a fixed $m_1 \times m_2$ matrix. This function is hemicontinuous where it is non empty (Theorem 14 in [47]). We can approach X by an increasing sequence of polyhedra X^t in the following sense:

$$X^t \subseteq X^{t+1} \subseteq X \text{ for all } t$$

and for all $x \in [q,p] \cap X$

$$x = \lim_{t \to \infty} x^t \text{ where } x^t \text{ is the projection of } x \text{ on } X^t$$

It is easy to check that lower hemicontinuity is preserved by (finite or infinite) union, as well as by the closure operation. As X is the closure of $\cup_t X^t$, so Φ^X is the closure of $\cup_t \Phi^{X^t}$, and we conclude that Φ^X is lower hemicontinuous.

10.2 Proposition 2

Fix $X = \{\sum_N x_i = \beta\} \cap C$ with C closed convex and fully symmetric, and let f be a rule meeting EFF, SYM, CONT, and SGSP. By Lemma 1 f is peak only.

Step 1 For any $p \in X_N$ such that $x = f(p)$, and any two agents labeled 1 such that $p_1 > p_2$, we claim that there is exactly three possible configurations of their allocations x_1, x_2:

$$p_1 > p_2 > x_1 = x_2 \text{ or } x_1 = x_2 > p_1 > p_2 \text{ or } p_1 \geq x_1 \geq x_2 \geq p_2$$

By uncompromisingness (Lemma 1) $f_1(x_1, p_2, p_{-1,2}) = x_1$. If $f_2(x_1, p_2, p_{-1,2}) \neq x_2$ then there is a preference $\succeq \in \mathcal{SP}(X_2)$ which is not indifferent between these two allocations: then coalition $\{1, 2\}$ has an opportunity to weakly misreport, which is impossible, so we conclude $x_1 = x_2$. The same argument applies for the cases $p_1 > p_2 > x_1$ and $x_i > p_1 > p_2$ for $i = 1, 2$. The remaining case is $x_1, x_2 \in [p_1, p_2]$ and we must exclude the configuration $p_1 \geq x_2 > x_1 \geq p_2$. By SYM the allocation $(x_2, x_1, x_{-1,2})$ is in X and by convexity of X so is $(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}, x_{-1,2})$: the latter is Pareto superior to $f(p)$, a contradiction.

Step 2 We fix an arbitrary profile p and define $N_- = \{i \in N | p_i < x_i\}$, $N_0 = \{i \in N | p_i = x_i\}$ and $N_+ = \{i \in N | p_i > x_i\}$. By Step 1 and SYM all i in N_- (resp. N_+) have the same allocation x_- (resp. x_+). Again by Step 1 and SYM for $j \in N_0$ and $i \in N_-$ inequality $p_j \leq x_-$ is impossible: so $x_- \leq p_j$ for all $j \in N_0$. A similar argument gives $p_j \leq x_+$.

We claim that $x \in X \cap [\omega, p]$. From $x_- \leq x_j \leq x_+$ for all $j \in N_0$ and $\sum_N x_i = \beta$ we see that $x_- \leq \omega_i = \frac{\beta}{n} \leq x_+$, therefore $p_i < x_- = x_i \leq \omega_i$ in N_-, and similarly $\omega_i \leq x_i = x_+ < p_i$ in N_+. Finally $x_i = p_i$ in N_0.

So the allocation x is entirely described by the two numbers x_+, x_-,where $\frac{\beta}{n} \leq x_+ \leq +\infty$ and $-\infty \leq x_- \leq \frac{\beta}{n}$. That is, if $p_i > x_+$ agent i gets x_+, she gets x_- if $p_i < x_-$, and she gets p_i if $x_- \leq p_i \leq x_+$. Note that $x_+ = +\infty$ (resp. $x_- = -\infty$) if and only if $N_+ = \varnothing$ (resp. $N_- = \varnothing$).

Now the equality $\sum_N x_i = \beta$ reduces to

$$|\{i : x_+ < p_i\}| \times (x_+ - \frac{\beta}{n}) + \sum_{i: \frac{\beta}{n} \leq p_i \leq x_+} (p_i - \frac{\beta}{n}) =$$

$$= |\{i : p_i < x_-\}| \times (\frac{\beta}{n} - x_-) + \sum_{i:x_- \leq p_i \leq \frac{\beta}{n}} (\frac{\beta}{n} - p_i) \quad (11)$$

Clearly the first term in the equality increases in x_+ while the second term decreases in x_-.

Step 3 We compare now $x = f(p)$ and $z = f^\omega(p)$. By Theorem 1 f^ω meets EFF, SYM, CONT, and SGSP just like f, therefore by Steps 1, 2 above, z is described by two numbers z_+, z_- just like x and they solve the same equation (11). By the monotonicity properties above, if $z \neq x$ we must have either $\{z_+ > x_+$ and $z_- < x_-\}$ or $\{z_+ < x_+$ and $z_- > x_-\}$. In the former case z is Pareto superior to x, and vice versa in the latter case. This is impossible because both rules are efficient.

10.3 Proposition 3

Statement i) We let the reader check that the argument detailed for example (4) applies as well to any convex, compact X symmetric and of dimension two; the shape of X inside X_{12} is the same, except when some of the four corners are actually feasible, but those cases are easy. Similarly if X is unbounded.

Statement ii) Here we choose a function θ_0 from $\mathbb{R}$ into $\mathbb{R}_+ = [0, +\infty[$ such that its restriction θ_- to $\mathbb{R}_-$ is a decreasing bijection to $\mathbb{R}_+$, and its restriction θ_+ to $\mathbb{R}_+$ is an increasing bijection to $\mathbb{R}_+$. The canonical example used in the construction of f^ω is $\theta_0(x) = |x|$.

For $z \in \mathbb{R}^N$ we write $\theta(z) = (\theta_0(z_i))_{i \in N}$. Fixing (N, X), ω and θ we define a new rule $f^{\omega,\theta}$ as follows

$$f^{\omega,\theta}(p) = x \overset{def}{\Longleftrightarrow} \{x \in X \cap [\omega, p] \text{ and } \theta(x - \omega) = \arg \max_{\Delta^\theta(\omega,p)} \succeq_{lxmin}\}$$

where

$$z \in \Delta^\theta(\omega, p) \overset{def}{\Longleftrightarrow} \{z = \theta(x - \omega) \text{ for some } x \in X \cap [\omega, p]\}$$

When $\theta_-(z) = \theta_+(-z)$ this definition is exactly the same as (2). Not so otherwise, because θ treats differently a move above the default ω_i and one below it.

Then we follow step by step the proof of the Theorem to show that $f^{\omega,\theta}$ meets precisely the same properties as f^ω. The desired conclusion follows because the set of functions θ such that θ_- is not the mirror image of θ_+ is of infinite dimension.

As the range of $X \cap [\omega, p]$ by $x \to \theta(x-\omega)$ is a compact set, $\succeq_{lxmin}$ reaches its maximum in $\Delta^{\theta}(\omega, p)$. To prove uniqueness (despite the fact that this range may not be convex) we mimick the argument in Step 1. Assume x, y are two maximizers, S, T are disjoints (we use the same notations as in Step 1) and set $a = \theta(x)^{*1} = \theta(y)^{*1}$: then for all $k \in N$ $a \leq \min\{\theta_0(x_k), \theta_0(y_k)\} < \max\{\theta_0(x_k), \theta_0(y_k)\}$ implying $\min_{k \in N} \theta_0(\frac{x+y}{2})_k > a$ and contradicting the optimality of x, y. Then S and T must intersect and the argument ends by dropping this coordinate and invoking the separability of $\succeq_{lxmin}$.

The proofs of EFF, SGSP, SYM and EF are exactly as in the Theorem, so we do not repeat them.

Continuity is not much harder. We restrict attention first to an arbitrary orthant Θ and to the vectors p such that $p - \omega \in \Theta$. Because θ treats differently positive and negative deviations from ω, we keep Θ an arbitrary orthant; on the other hand normalizing ω to zero is without loss of generality. We set $h(p) = \theta(f^{\omega,\theta}(p))$ and prove first that $h(\cdot)^*$ is continuous. As $\theta(x)^{*1}$ is continuous Berge's theorem tells us that $h(p)^{*1} = \max\{\theta(x)^{*1} | x \in \Phi(0, p)\}$ is continuous as well. For the next coordinate we can write

$$h(p)^{*2} = \max\{\theta(x)^{*2} | x \in \Phi(0, p) \text{ and } \theta(x) \geq h(p)^{*1} e^N\}$$
$$= \max\{\theta(x)^{*2} | x \in \Phi(\theta_0^{-1}(h(p)^{*1}), p)\}$$

therefore Berge theorem applies again, and $h(\cdot)^{*2}$ is continuous. And so on as in the above proof.

Once $h(\cdot)^*$ is continuous, we take a converging sequence $p^t \to p$ as before and w a limit point of $f(p^t)$, i. e., $w = \lim_{t'} f(p^{t'})$ for some subsequence t' of t (omitting the superscripts). Then $\theta(f(p^{t'}))^* \to \theta(w)^*$ because θ and $x \to x^*$ are continuous; and $\theta(f(p^t))^* \to \theta(f(p))^*$ by the continuity of $h(\cdot)^*$. Thus $\theta(f(p))^* = \theta(w)^*$ and $w \in \Phi(0, p)$ by the hemicontinuity of Φ. We conclude $w = f(p)$ as was to be proved.

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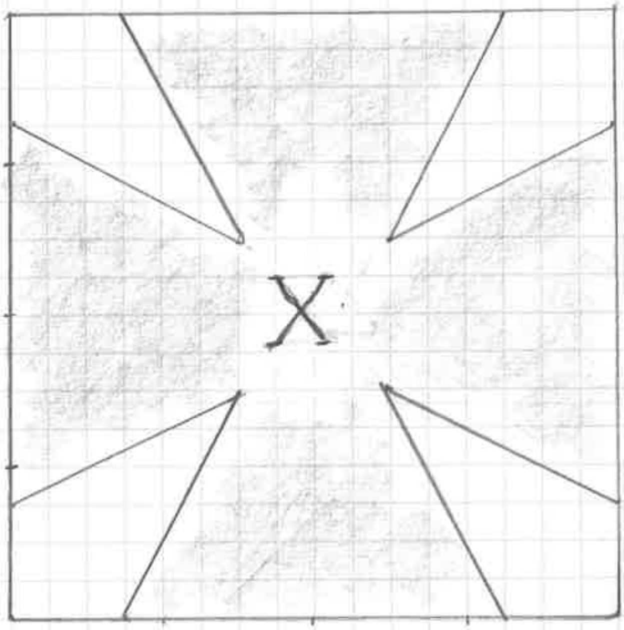
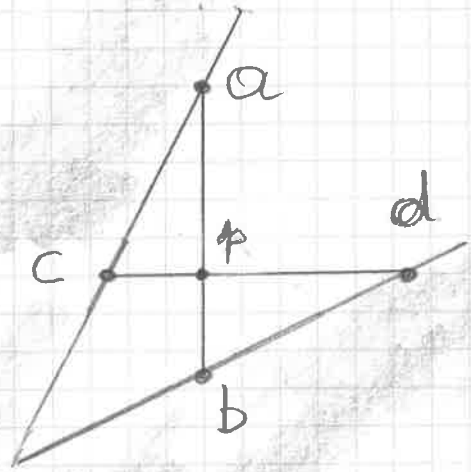


Fig 1



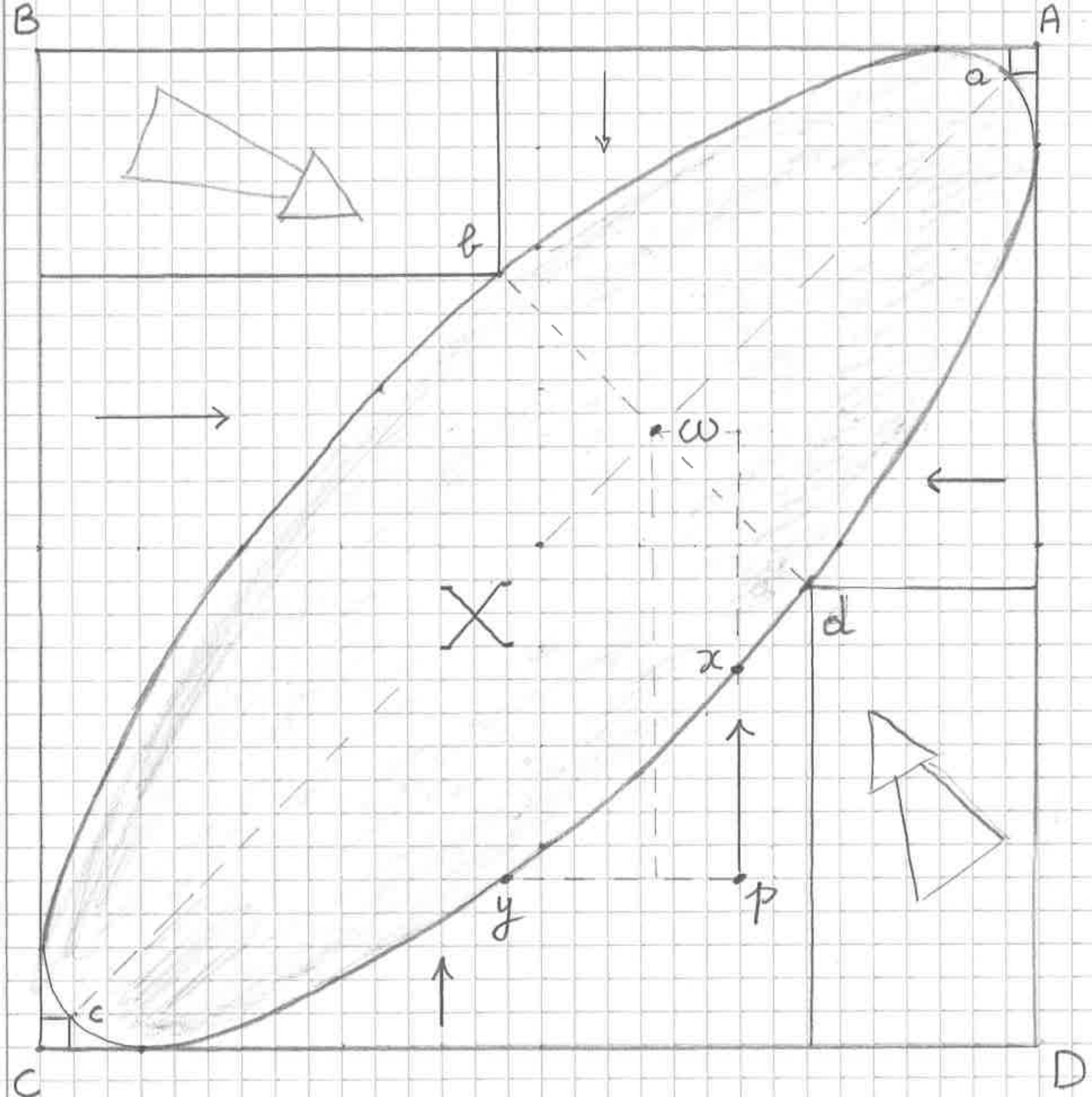


FIG 2

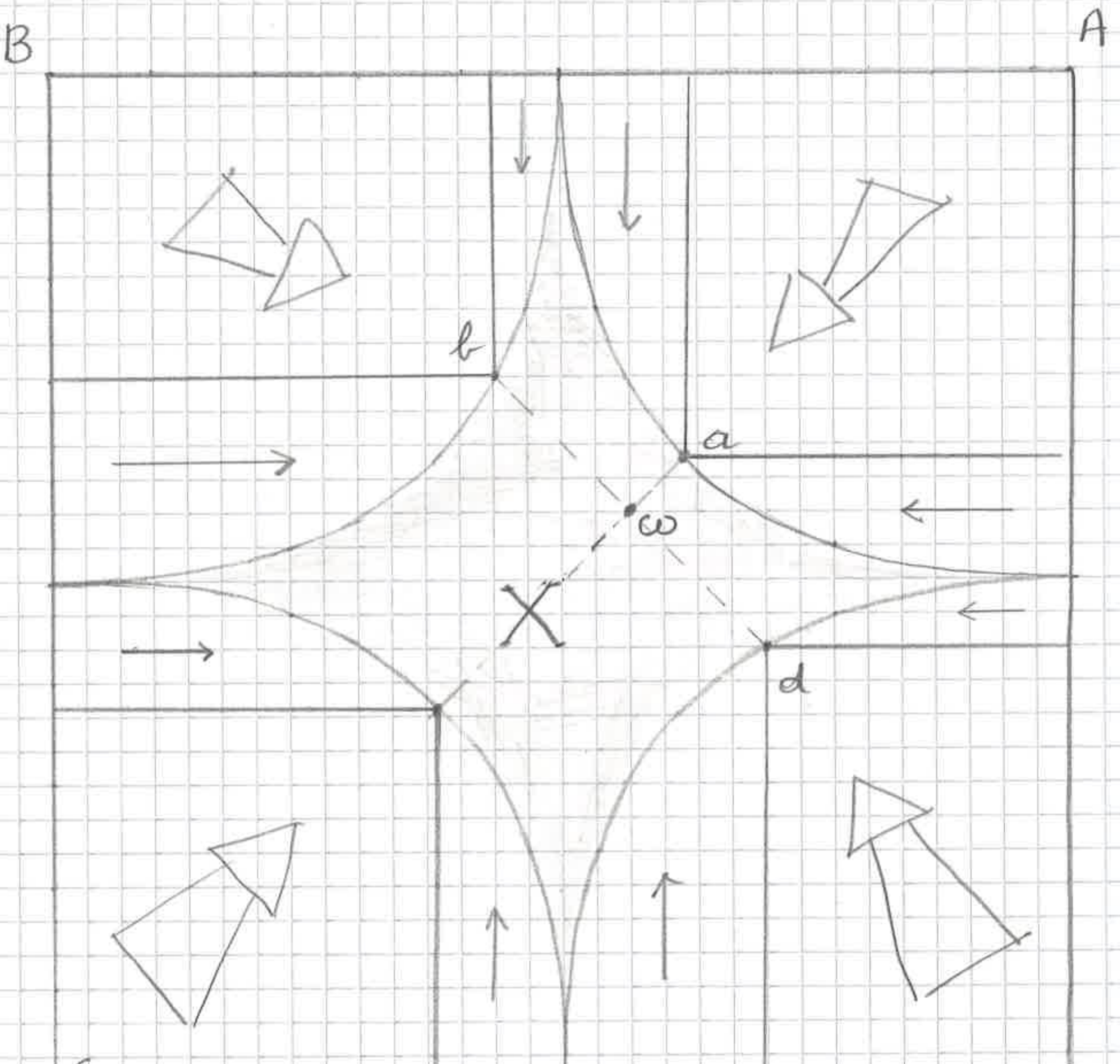


FIG 3