Computing Balanced Solutions for Large International Kidney Exchange Schemes When Cycle Length Is Unbounded

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Abstract

In kidney exchange programmes (KEPs) patients may swap their incompatible donors leading to cycles of kidney transplants. Countries try to merge their national patient-donor pools leading to international KEPs (IKEPs). How can we ensure long-term stability of an IKEP to prevent countries leaving? Long-term stability of an IKEP can be achieved through a credit-based system. The goal is to find, in each round, an optimal solution that closely approximates this target allocation. We provide both theoretical and experimental results for the case where the cycle length is unbounded.

Kidney Exchange

- We model a pool of patient-donor pairs as a directed graph $G = (V, A)$ (the ***compatibility graph***) in which V consists of the patient-donor pairs, and A consists of every arc (u, v) such that the donor of u is compatible with the patient of v.
- In a directed cycle $C = u_1u_2 \dots u_ku_1$, for some $k \geq 2$, the kidney of the donor of u_i could be given to the patient of u_{i+1} (with $u_{k+1} := u_1$). This is a ***k-way exchange*** using the *exchange cycle* C.
- An ***ℓ-cycle packing*** of G is a set $\mathcal{C}$ of directed cycles, each of length at most ℓ, that are pairwise vertex-disjoint.
- A solution for round r is an ℓ-cycle packing in the associated compatibility graph G^r. To help as many patients as possible in each round r, we seek an ***optimal* solution**, i.e., a *maximum (size)* ℓ-cycle packing of G^r.

Theorem [1]: If $\ell = 2$ or $\ell = \infty$, we can find an optimal solution for a KEP round in polynomial time; else this is NP-hard.

International Kidney Exchange

As merging pools of national KEPs leads to better outcomes, *international* KEPs (IKEPs) are formed [5, 7]. *How can we ensure long-term stability of an IKEP to prevent countries leaving?*

- A ***(cooperative) game*** is a pair (N, v), where N is a set of n *players* and $v : 2^N \to \mathbb{R}$ is a *value function* with $v(\emptyset) = 0$. A subset $S \subseteq N$ is a *coalition*. If for every possible partition $(S_1, \dots, S_r)$ of N it holds that $v(N) \geq v(S_1) + \dots + v(S_r)$, then players will benefit most by forming the *grand coalition* N.
- An ***allocation*** is a vector $x \in \mathbb{R}^N$ with $x(N) = v(N)$ (we write $x(S) = \sum_{p \in S} x_p$ for $S \subseteq N$).
- A ***solution concept*** prescribes a set of fair allocations for a game (N, v). In particular, the **core** allocations ensure N is stable. However, a core may be empty.
  partitioned ℓ-permutation game on a directed graph $G = (V, A)$ with a partition $(V_1, \dots, V_n)$ of V is the game (N, v), where $N = \{1, \dots, n\}$, and for $S \subseteq N$, the value $v(S)$ is the maximum size of an ℓ-cycle packing of $G[\bigcup_{i \in S} V_i]$.
- We obtain a ***partitioned matching game*** [2, 4] if $\ell = 2$, and a ***partitioned permutation game*** if $\ell = \infty$. The ***width*** of (N, v) is $c = \max\{|V_i| \mid 1 \leq i \leq n\}$.

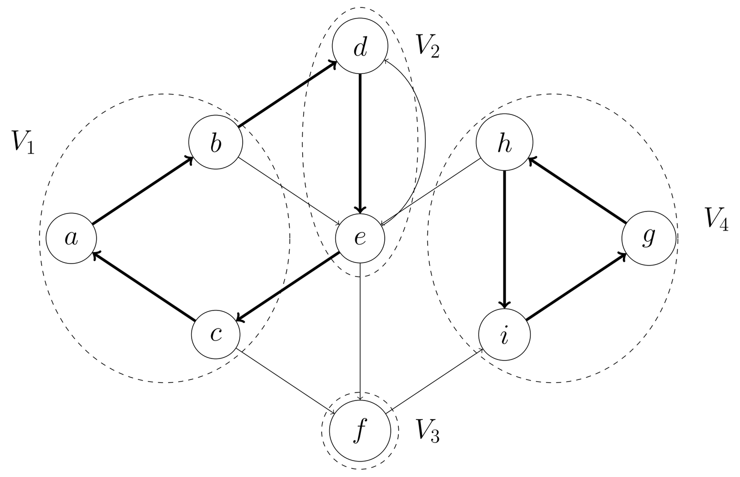

Figure 1. A permutation game (N, v) of width $c = 3$ defined on a graph $G = (V, A)$. Note that $N = \{1, 2, 3, 4\}$ and $V = V_1 \cup V_2 \cup V_3 \cup V_4$ with $V_1 = \{a, b, c\}$, $V_2 = \{d, e\}$ and $V_3 = \{f\}$ and $V_4 = \{g, h, i\}$, as indicated by the dotted circles. Also note that $v(N) = v(\{1, 2, 4\}) = 8$, that is the size of cycle packing $\mathcal{C} = \{abdeca, high\}$ as indicated by the thick edges, while $v(\{2\}) = 2$ and $v(\{4\}) = 3$ and $v(\{1\}) = v(\{3\}) = 0$.

To ensure IKEP stability, we use the model of Klimentova et al. [6], which is a ***credit-based*** system.

- In each round, countries are given an initial allocation (as prescribed solution concepts) of the total number of kidney transplants. This allocation is adjusted by a credit function yielding a target allocation.

We must also determine how to choose in each round r a maximum ℓ-cycle packing $\mathcal{C}$ (optimal solution) of the corresponding compatibility graph G.

- $\mathcal{C}$ is ***strongly close*** to x if $d(\mathcal{C})$ is lexicographically minimal over all optimal solutions.
- If we only minimize $d_1(\mathcal{C}) = \max_{p \in N}\{|x_p - s_p(\mathcal{C})|\}$ over all optimal solutions, we obtain a ***weakly close*** optimal solution.

Related Work

- It is NP-hard to find a weakly close maximum matching even for $|N| = 2$ once the games are defined on edge-weighted graphs [4].

Theorem [3] For partitioned matching games, the problem of finding an optimal solution that is strongly close to a given target allocation x is polynomial-time solvable.

Theoretical Results

Theorem The core of every partitioned permutation game is non-empty, and it is possible to find a core allocation in polynomial time. Moreover, for partitioned permutation games of fixed width c, the problem of deciding if an allocation is in the core is polynomial-time solvable if $c = 1$ and coNP-complete if $c \geq 2$.

Theorem For partitioned permutation games even of width 2, the problem of finding an optimal solution that is weakly or strongly close to a given target allocation x is NP-hard.

Theorem For a partitioned permutation game (N, v) on a directed graph $G = (V, A)$, the problem of finding an optimal solution that is weakly or strongly close to a given target allocation x can be solved by a randomized algorithm in $O(|A|^{O(n)})$ time.

Simulation Results

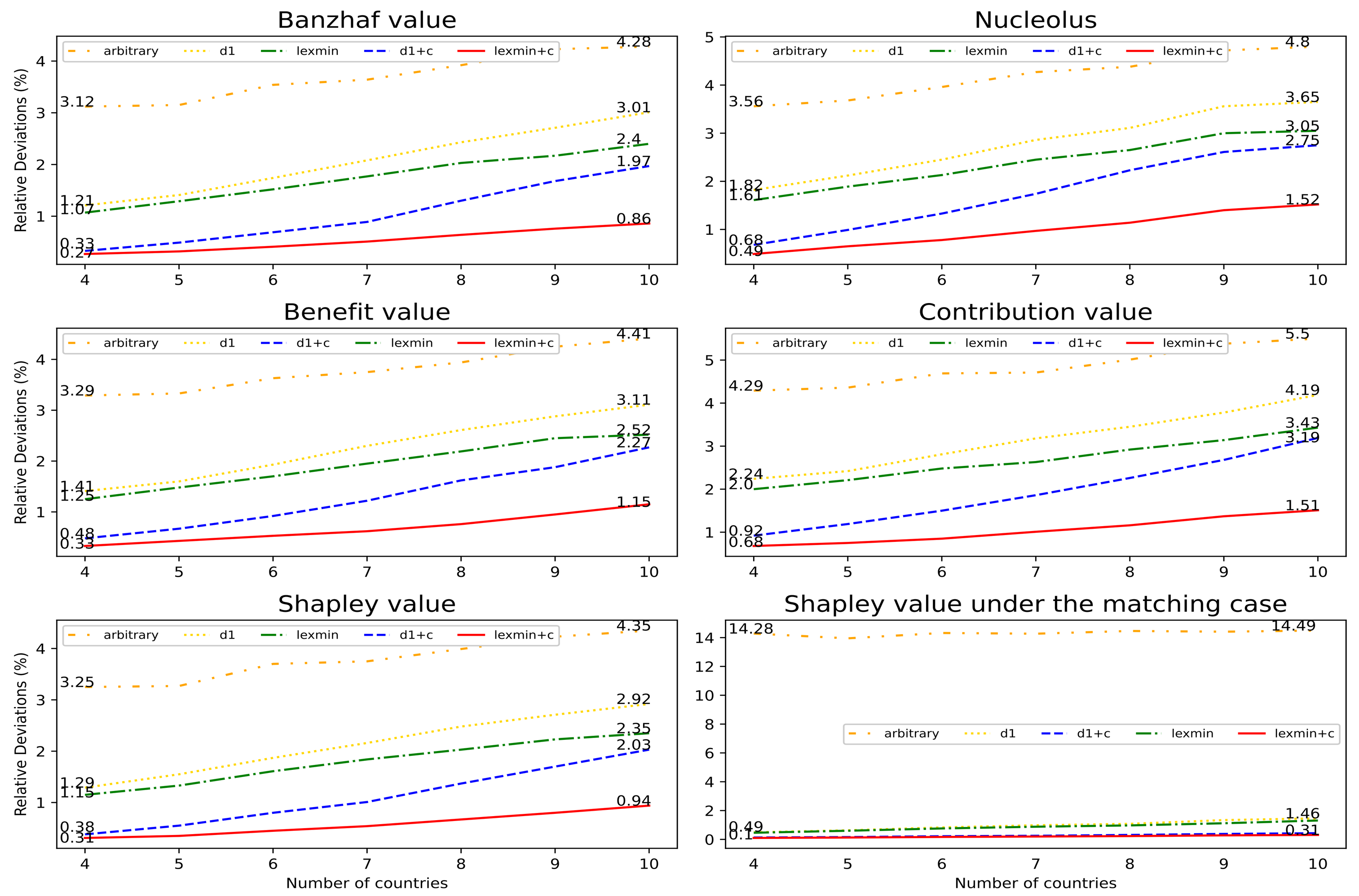

Figure 2. Average total relative deviations for each of the five solution concepts under the five different scenarios, where the number of countries n ranges from 4 to 10. For comparison, the bottom right figure displays a result from [3] for $\ell = 2$, namely for the Shapley value, which behaved best for $\ell = 2$.

- A credit system using strongly close optimal solutions makes an IKEP the most balanced. without decreasing the overall number of transplants
- The Banzhaf value yields the best results: on average, a deviation of up to 0.90% from the target allocation.
- Moving from $\ell = 2$ to $\ell = \infty$ yields on average 46% more kidney transplants, but cycles may be very large, in particular in the starting round.

References

[1] Abraham, D.J., Blum, A., Sandholm, T.: Clearing algorithms for barter exchange markets: enabling nationwide kidney exchanges. Proceedings EC 2007 pp. 295–304 (2007)

[2] Benedek, M., Biró, P., Johnson, M., Paulusma, D., Ye, X.: The complexity of matching games: A survey. Journal of Artificial Intelligence Research **77**, 459–485 (2023)

[3] Benedek, M., Biró, P., Paulusma, D., Ye, X.: Computing balanced solutions for large international kidney exchange schemes. arXiv preprint arXiv:2109.06788 (2021)

[4] Biró, P., Kern, W., Pálvölgyi, D., Paulusma, D.: Generalized matching games for international kidney exchange. Proceedings AAMAS 2019 pp. 413–421 (2019)

[5] Böhmig, G.A., Fronek, J., Slavcev, A., Fischer, G.F., Berlakovich, G., Viklicky, O.: Czech-Austrian kidney paired donation: first european cross-border living donor kidney exchange. Transplant International **30**, 638–639 (2017)

[6] Klimentova, X., Viana, A., Pedroso, J.a.P., Santos, N.: Fairness models for multi-agent kidney exchange programmes. Omega **102**, 102333 (2021)

[7] Valentín, M.O., Garcia, M., Costa, A.N., Bolotinha, C., Guirado, L., Vistoli, F., Breda, A., Fiaschetti, P., Dominguez-Gil, B.: International cooperation for kidney exchange success. Transplantation **103**, 180–181 (2019)