Dynamic online matching with budget refills

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Motivation: user allocation

Fig. 1: ad allocations

- Users arrive sequentially on websites.
- Each ad campaign is displayed for a specific period of time.
- The goal is to match users with ads based on their preferences.

Matching on bipartite graph

Offline version:
Let $G = (U, V, E)$ be a a bipartite graph:

- U and V two sets of vertices.
- Each node in U has a budget $b_u = 1$.
- Edges are **only between** U and V, $E = \{(u, v), u \in U, v \in V\}$.

Online version: For $t = 1, \dots, |V|$:

- v_t arrives with its edges.
- The algorithm can match it to a free vertex in U.
- The matching decision is irrevocable.

The performance measure of an algorithm

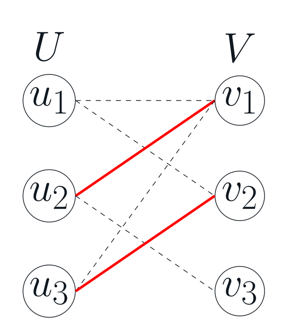


Fig. 2: $ALG = 2$

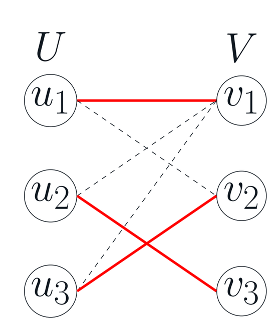


Fig. 3: $OPT = 3$

The competitive ratio: for $G \in \mathcal{G}$, where $\mathcal{G}$ is a family of graphs, the competitive ratio is defined as:

$$\mathtt{CR} = \frac{\mathbb{E}(ALG(G))}{OPT(G)}$$

Note that $0 \le \mathtt{CR} \le 1$.

Related works

Online matching with unitary budget:

$\mathtt{Greedy}$ **algorithm**: For $t = 1, \dots, |V|$:
Match v_t to any free neighbor at random.
Performance of $\mathtt{Greedy}$**:** In the Adversarial setting, for $\mathtt{Greedy}$ (and any deterministic alg.)

$$\mathtt{CR}(\mathtt{Greedy}) = \frac{1}{2}$$

A randomized algorithm can achieve,

$$\mathtt{CR}(\mathtt{ALG}) \ge 1 - \frac{1}{e} \approx 0.63$$

Online b-matching problem: In this setting $b_u = b > 1$

$\mathtt{Balance}$ **algorithm**: For $t = 1, \dots, |V|$:
Match v_t to a neighbor with highest remaining budget.
Performance of $\mathtt{Balance}$**:** [2], when $b_u = b$ for all $u \in U$,

$$\mathtt{CR}(\mathtt{Balance}) = 1 - \frac{1}{(1 + 1/b)^b}$$

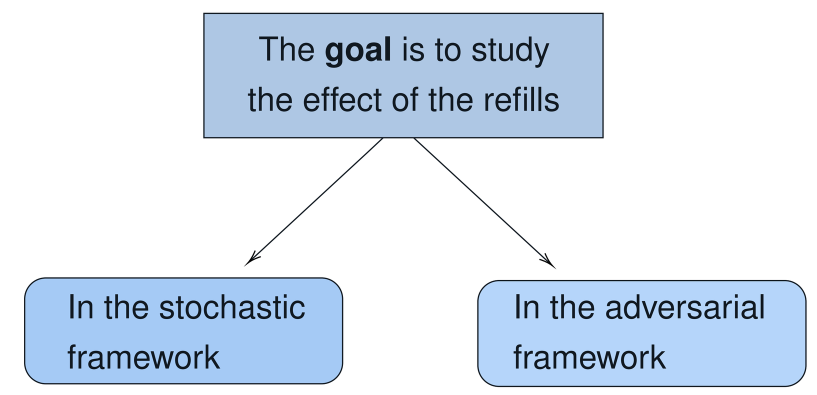
[1] with different budget b_u,

$$\mathtt{CR}(\mathtt{Balance}) = 1 - \frac{1}{(1 + 1/b_{\min})^{b_{\min}}}, \quad \text{with } \min_{u \in U} b_u$$

More realistic setting: online matching with budget refills

Let $G \in \mathcal{G}$, with $G = (U, V, E)$ a bipartite graph,

- $|U| = n, |V| = T$ with $T \ge n$. Nodes in U are offline and nodes in V are revealed sequentially.
- Each node in U has a budget $b_{u,t} \ge 0$, at time $t \in [T]$.



Stochastic framework

$\mathcal{G}$ is a family of Erdős–Rényi sparse random graphs:

- Edges occurring independently with probability $p = a/n$. Each node in U has a budget $b_{u,t} \in \mathbb{N}$:

$$b_{u,t} = \min(K, b_{u,t-1} - x_{u,t} + \eta_t) \tag{1}$$

  η_t is a realization of a Bernoulli random variable $\mathcal{B}(\frac{\beta}{n})$, $x_{u,t} = 1$ if u and t are matched and $x_{u,t} = 0$ if not.

Theorem (first result): For $\psi = \frac{T}{n} \ge 1$, with high probability $\mathtt{Greedy}(G, T)$ is given by,

$$\mathtt{Greedy}(G, T) = nh(\psi) + o(n)$$

where $h(\tau)$ is solution of the following system denoted (A),

$$\begin{cases} \dot{h}(\tau) = 1 - e^{-a(1 - z_0(\tau))} & 1/n \le \tau \le \psi \\ \dot{z}_0(\tau) = -z_0(\tau)\beta + \frac{z_1(\tau)}{1 - z_0(\tau)}(1 - e^{-a + az_0(\tau)}) & \text{for } k = 0 \\ \dot{z}_k(\tau) = (z_{k-1}(\tau) - z_k(\tau))\beta + (z_{k+1}(\tau) - z_k(\tau))\frac{1 - e^{-a + az_0(\tau)}}{1 - z_0(\tau)} & \text{for } 1 \le k \le K - 1 \\ \dot{z}_k(\tau) = \beta\, z_{k-1}(\tau) - z_k(\tau)\frac{1 - e^{-a(1 - z_0(\tau))}}{1 - z_0(\tau)} & \text{for } k = K \\ \sum_{k=0}^{K} z_k(\tau) = 1 \end{cases}$$

Corollary: For $K \ge 1$, with probability at least $1 - 2e^{-a^2 n^{\frac{3}{2}}/8T}$,

$$|\mathtt{Greedy}(G, T) - nh^*(T/n)| \le o(T)$$

with $h^*(x) = \int_{1/n}^{x} (1 - e^{-a(1 - z_0^*)}) \mathrm{d}\tau = \left(x - \frac{1}{n}\right)(1 - e^{-a(1 - z_0^*)})$, and z_0^* is the unique solution of $\sum_{k=0}^{K} z_0^* \left(\frac{\beta}{g(z_0^*)}\right)^k = 1$ with $g(z_0^*) = \frac{1 - e^{-a(1 - z_0^*)}}{1 - z_0^*}$.

Corollary: For $K = 1$, with probability at least $1 - 2e^{-a^2 n^{\frac{3}{2}}/8T}$,

$$\left|\mathbb{E}[\mathtt{Greedy}(G, T)] - T(1 - e^{-a(1 - z_0^*)})\right| \le c\frac{T}{(\log(T))^{3/4}} = o(T)$$

where $z_0^* = \frac{1}{\beta} - \frac{1}{a} W\left(\frac{a}{\beta} e^{-a\left(1 - \frac{1}{\beta}\right)}\right)$, with $W(\cdot)$ the Lambert function, and c is some universal constant.

Proposition (informal): For $T, K, n, b_0, \beta \in \mathbb{N}^*$,

$$\mathtt{CR}^{\mathrm{sto}}(\mathtt{Greedy}, \mathcal{D}) \ge \frac{Tg(z_0^*)(1 - z_0^*) + nb_0 - n\left(\frac{\beta}{g(z_0^*) - \beta} - \frac{(K+1)\beta^{K+1}}{g(z_0^*)^{K+1} - \beta^{K+1}}\right)}{nb_0 + \beta T} + \mathcal{O}_{K,\beta}(T^{-1/4})$$

Theorem: For any $\alpha, \beta > 0$, the competitive ratio tends to 1, as T, K, n approach infinity, as

$$\lim_{K, n \to +\infty} \lim_{T \to +\infty} \mathtt{CR}^{\mathrm{sto}}(\mathtt{Greedy}, \mathcal{D}) = 1$$

Adversarial framework

$G = (U, V, E)$ is a bipartite graph generated by an oblivious adversary:

- $|U| = n$ and $|V| = T$, with $T \ge n$.
- Each node in U has a budget $b_{u,t} \in \mathbb{N}$:

$$b_{u,t} = b_{u,t-1} - x_{u,t} + \mathbb{1}_{t \bmod m = 0} \tag{2}$$

Theorem (informal): For $m \ge \sqrt{T}$,

$$\mathtt{CR}(\mathtt{Balance}) \le 1 - \frac{1}{\left(1 + \frac{1}{b_0}\right)^{b_0}}$$

Theorem (informal): For $m = o(\sqrt{T})$,

$$\mathtt{CR}(\mathtt{Balance}) \le 1 - \underbrace{\frac{(1 - \alpha)}{e^{(1 - \alpha)}}}_{\simeq 0.73325\ldots} \tag{3}$$

where α is defined by $\frac{1}{2} = \int_0^{\alpha} \frac{xe^x}{1 - x} \mathrm{d}x$.

$\mathtt{Balance}$ **is the optimal deterministic algorithm**

Theorem (informal):

$$\sup_{\mathtt{ALG}} \inf_{G \in \mathcal{G}} \mathtt{CR}(\mathtt{ALG}) \le \inf_{G \in \mathcal{G}} \mathtt{CR}(\mathtt{Balance}) \tag{4}$$

References

[1] Susanne Albers and Sebastian Schubert. "Optimal Algorithms for Online b-Matching with Variable Vertex Capacities". In: *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2021)*. Ed. by Mary Wootters and Laura Sanità. Vol. 207. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2021, 2:1–2:18.

[2] Bala Kalyanasundaram and Kirk R. Pruhs. "An optimal deterministic algorithm for online b-matching". In: *Theoretical Computer Science* 233.1 (2000), pp. 319–325.