The Computational Complexity of the Housing Market

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Motivation

Trade of indivisible goods between unit-demand agents with money. Introduced by Quinzii (1984), Gale (1984), and Svensson (1984).

- **House allocation problem** without money solved by Gale's Top Trading Cycle algorithm.
- Housing market with **quasilinear utilities** (in money) solved, e.g., by Hungarian algorithm.

For high-value items (e.g., houses):

- agents experience income effects,
- willingness to pay depends on level of wealth.

Nothing known about computational complexity of 'more realistic' assumptions, e.g. **'soft' budget constraints** or **costs of borrowing**.

Gale's Housing Market

- **Agents** $\{1,\dots,n\}$ and **houses** $\{1,\dots,n\}$.
- Agent i has **preference sets** $P_0^i, P_1^i, \dots, P_n^i$ covering $\mathbb{R}^n$.
- Demands **house** j at prices $\boldsymbol{p} \in P_j^i$, and **nothing** at $\boldsymbol{p} \in P_0^i$.
- Indifferent between houses j and k if $\boldsymbol{p} \in P_j^i \cap P_k^i$.

competitive equilibrium: Prices $\boldsymbol{p} \in \mathbb{R}^n$ and envy-free allocation $\pi : [n] \to [n]$ of houses to agents: $\boldsymbol{p} \in P_{\pi(i)}^i$.

ε-approximate competitive equilibrium: Prices $\boldsymbol{p} \in \mathbb{R}^n$ and allocation $\pi : [n] \to [n]$ of houses to agents s.t. $\boldsymbol{p}$ is ε-close to $P_{\pi(i)}^i$.

Gale's assumptions

(i) P_j^i are closed.
(ii) For each house j: $\boldsymbol{p} \notin P_j^i$ if $p_j \geq 1$.
(iii) $P_1^i, \dots, P_n^i$ cover lower faces of n-dimensional unit cube.

Theorem: (Gale, 1984) Competitive equilibrium exists if agents satisfy Gale's assumptions.

Theorem: ε-approximate competitive equilibrium exists if agents satisfy Gale's assumptions (ii) and (iii).

Computational Questions

How hard is it to compute (approximate) competitive equilibria?

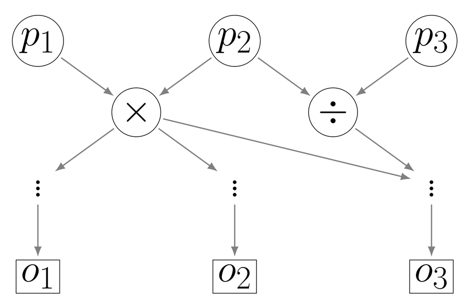
Housing: Given agent preferences that satisfy Gale's assumptions, compute equilibrium $(\boldsymbol{p}, \pi)$.

ε-Housing: Given $\varepsilon > 0$ and agent preferences that satisfy Gale's assumptions (ii) and (iii), compute ε-equilibrium $(\boldsymbol{p}, \pi)$.

Computational representation of preference sets: $P_0, P_1, \dots, P_n$

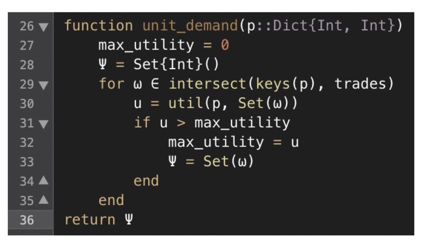
Arithmetic circuits

- Input: prices $\boldsymbol{p} \in \mathbb{R}^n$.
- Output: $\boldsymbol{o} \in \mathbb{R}^{n+1}$ with $o_j = 0$ iff $\boldsymbol{p} \in P_j$.
- Base $\{+, -, \times, \div, \min, \max\}$, rational constants.



Polynomial-time algorithms

- Input: prices $\boldsymbol{p} \in \mathbb{R}^n$.
- Output: Houses demanded at $\boldsymbol{p}$.
- Polynomial runtime guarantees.



Function oracles $f_1, \dots, f_n$.

- Input: prices $\boldsymbol{p} \in \mathbb{R}^n$.
- Output: $f_j(\boldsymbol{p}) = 1$ iff $\boldsymbol{p} \in P_j$.
- Similar to demand queries in auctions

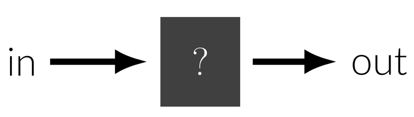


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Illustration with Two Houses

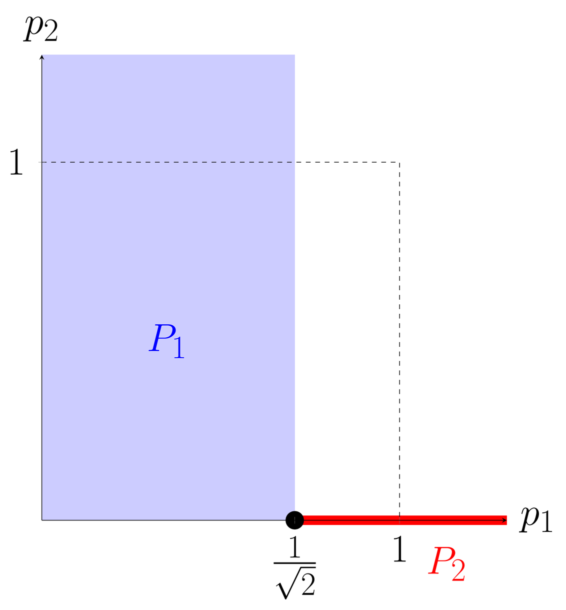
Two agents with identical preferences

$P_0 = \mathbb{R}^n$.

$P_1 = \{\boldsymbol{p} \in \mathbb{R}^n \mid 0 \leq p_1 \leq \frac{1}{\sqrt{2}}\}$.

$P_2 = \{\boldsymbol{p} \in \mathbb{R}^n \mid p_1 \geq \frac{1}{\sqrt{2}}$ and $p_2 = 0\}$.

Unique equilibrium prices at $\boldsymbol{p} = (\frac{1}{\sqrt{2}}, 0)$



Results

How hard is it to find a competitive equilibrium?

Theorem 1: Finding an exact competitive equilibrium in the circuit model is FIXP-complete.

Theorem 2: Finding an approximate equilibrium in the algorithm or query model takes a polynomial number of queries / steps if the market consists of two agents.

Theorem 3: Finding an approximate equilibrium in the algorithm model is PPAD-complete, even if we restrict the market to three identical agents.

Theorem 4: Finding an approximate equilibrium in the query model takes exponentially many query in the approximation parameter for four or more agents.

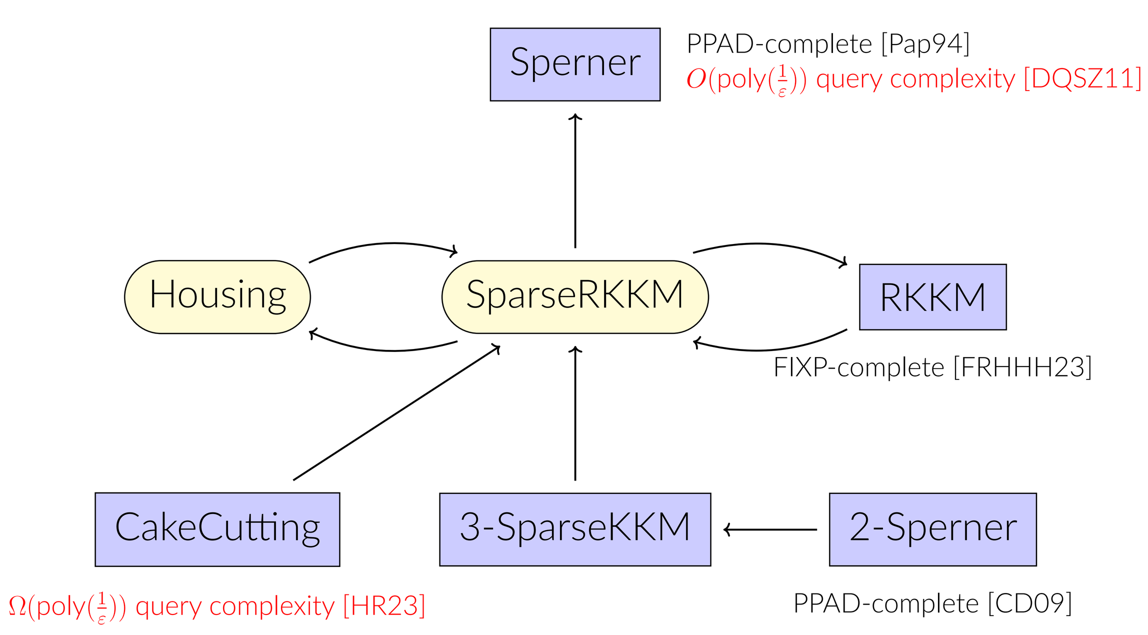
Implication: We can't find competitive equilibria efficiently.

Complexity Classes

PPAD: approximate Nash equilibrium, approximate market equilibria in Arrow-Debreu and Fisher markets, cake cutting, Sperner, End-of-the-Line.

FIXP: (Exact) Nash equilibrium, KKM, Rainbow-KKM, etc.

Proof Overview



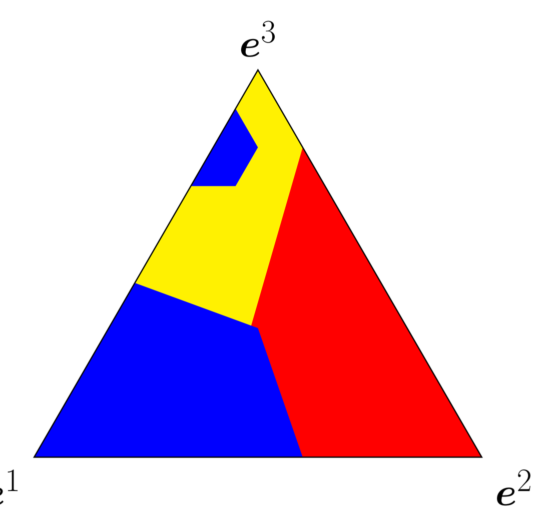
(Rainbow-)KKM

Standard simplex $\Delta_{n-1} := \text{conv}\{\boldsymbol{e}^j \mid j \in [n]\}$.

F_S is the face of Δ_{n-1} spanned by vertices $\{\boldsymbol{e}^j \mid j \in S\}$.

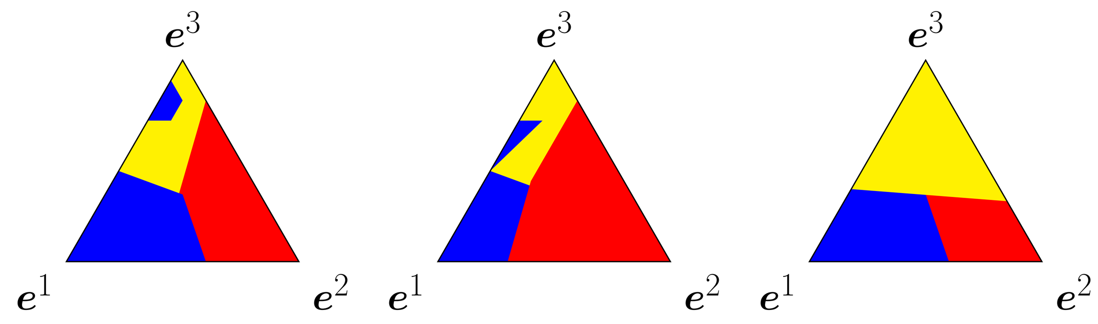
KKM covering: $C_1, \dots, C_n \subseteq \Delta_{n-1}$ s.t. $F_S \subseteq \bigcup_{j \in S} C_j$ for all $S \subseteq [n]$.

Sparse KKM covering: For every $j \in [n]$, $\boldsymbol{x} \in C_j$ implies $x_j > 0$.



(Approximate) KKM Lemma (cf. Knaster et al., 1929):

- For any KKM covering $C_1, \dots, C_n$ of closed sets, there exists $\boldsymbol{x}$ contained in every C_j.
- For any KKM covering $C_1, \dots, C_n$ and $\varepsilon > 0$, there exists $\boldsymbol{x}$ ε-close to each C_j.



(Approximate) Rainbow-KKM Lemma (cf. Gale, 1984):

- Any collection of n KKM coverings $C_1^i, \dots, C_n^i$ with closed sets admits a point $\boldsymbol{x}$ and permutation π of $[n]$ so that $\boldsymbol{x}$ lies in every $C_{\pi(i)}^i$.
- Any collection of n KKM coverings $C_1^i, \dots, C_n^i$, admits a point $\boldsymbol{x}$ and permutation π of $[n]$ so that $\boldsymbol{x}$ is ε-close to every $C_{\pi(i)}^i$.

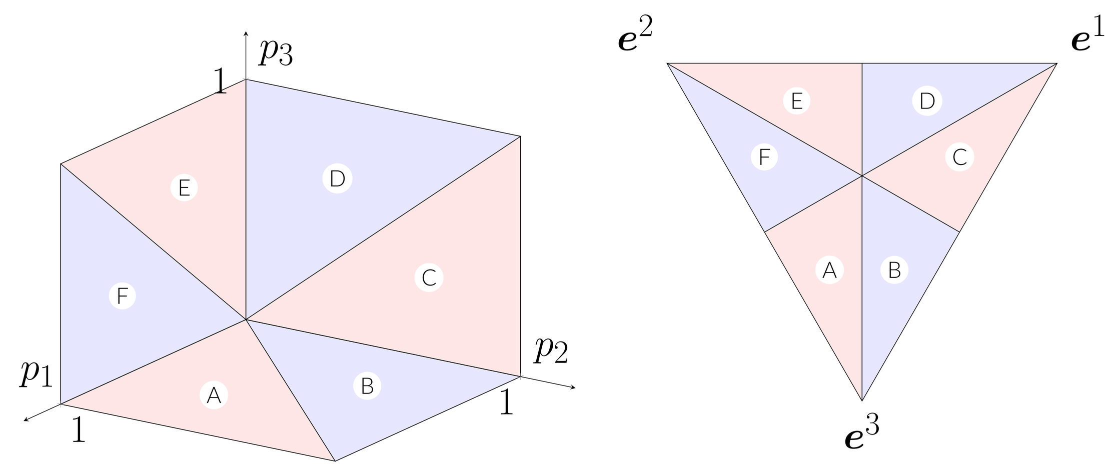
The Equivalence of Housing and Sparse RKKM

Gale (1984) defines **homeomorphism** ϕ.

Divide domain of housing market into $n!$ simplices corresponding to permutations π of $[n]$.

$$\Sigma_\pi := \{\boldsymbol{p} \in \Sigma_n \mid p_{\pi(1)} \geq p_{\pi(2)} \geq \dots \geq p_{\pi(n)} = 0\}$$

$$\phi(\boldsymbol{p})_{\pi(k)} := \frac{1 - p_{\pi(1)}}{n} + \frac{p_{\pi(1)} - p_{\pi(2)}}{n-1} + \frac{p_{\pi(k-1)} - p_{\pi(k)}}{n-k+1}.$$

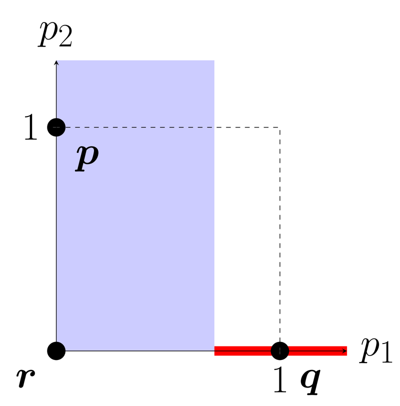


The Reductions

	Housing		SparseRKKM
Instance	$(P_1^i, \dots, P_n^i)_{i \in [n]}$	$\Rightarrow$	$(\phi(P_1^i), \dots, \phi(P_n^i))_{i \in [n]}$
Solution	$\phi^{-1}(\boldsymbol{x}), \pi$	$\Leftarrow$	$\boldsymbol{x}, \pi$

Lemma: ϕ is n-Lipschitz and ϕ^{-1} is n^2-Lipschitz, which gives reductions for approximation versions.

Two-Agent Markets



Algorithm 1 Binary search

Let $\boldsymbol{p} \leftarrow (0,1) \in P_1^1$, $\boldsymbol{q} \leftarrow (1,0) \in P_2^2$ and $\boldsymbol{r} \leftarrow (0,0)$.
while $\|\boldsymbol{p} - \boldsymbol{q}\| > \varepsilon$ and $\boldsymbol{r} \in P_1^1 \cup P_2^2$ **do**
  Update $\boldsymbol{p} \leftarrow \boldsymbol{r}$ if $\boldsymbol{r} \in P_1^1$ and $\boldsymbol{q} \leftarrow \boldsymbol{r}$ else.
  $\boldsymbol{r} \leftarrow \frac{1}{2}(\boldsymbol{p} + \boldsymbol{q})$.
return ε-equilibrium prices $\boldsymbol{r}$.

Outlook

We initiate the study of housing markets from the perspective of computational complexity.

We hope these results will stimulate further examination of the complexity of markets with income effects:

- What is the complexity of Housing with utility functions instead of preference sets?
- Does hardness continue to hold under the natural assumptions of monotonicity in money and no externalities as in (Quinzii, 1984; Svensson, 1984)?
- How might CakeCutting with monotonic valuations (Deng et al., 2012; Hollender & Rubinstein, 2023) map to assumptions for Housing?

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