Propositions as Sessions

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Kohei Honda, 1959–2012

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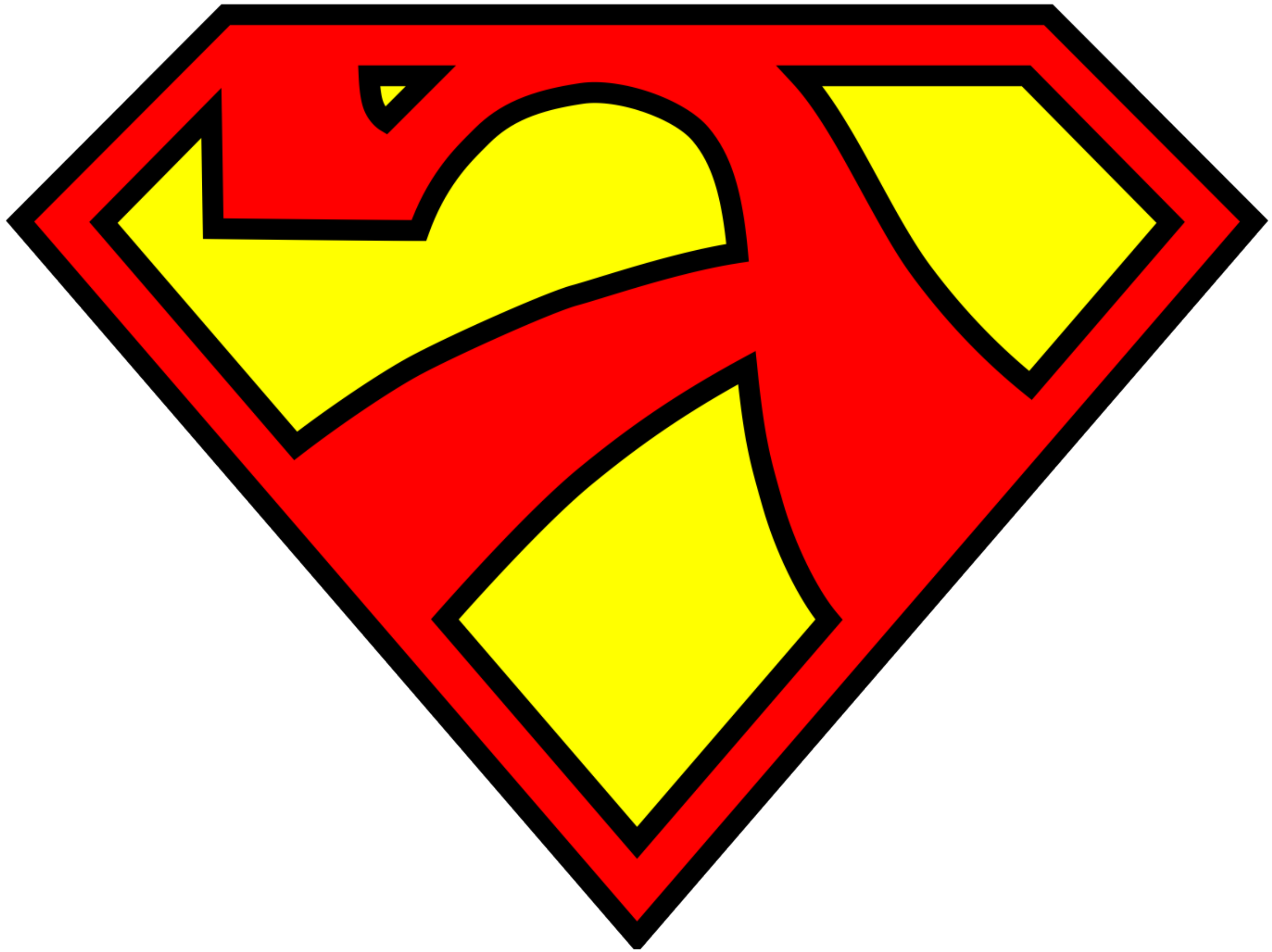
From Data Types to Session Types: A Basis for Concurrency and Distribution (ABCD)

Simon Gay, Nobuko Yoshida, Philip Wadler



Propositions as Types

propositions	*as*	types
proofs	*as*	programs
normalisation of proofs	*as*	evaluation of programs

Propositions as Types is robust

propositions	*as*	types
proofs	*as*	programs
normalisation of proofs	*as*	evaluation of programs

Intuitionistic Natural Deduction	$\leftrightarrow$	Simply-Typed Lambda Calculus
Quantification over propositions	$\leftrightarrow$	Polymorphism
Quantification over individuals	$\leftrightarrow$	Dependent types
Modal Logic	$\leftrightarrow$	Monads (state, exceptions)
Classical-Intuitionistic Embedding	$\leftrightarrow$	Continuation Passing Style

…but there's a missing link

propositions	*as*	types
proofs	*as*	programs
normalisation of proofs	*as*	evaluation of programs

Intuitionistic Natural Deduction	$\leftrightarrow$	Simply-Typed Lambda Calculus
Quantification over propositions	$\leftrightarrow$	Polymorphism
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???	$\leftrightarrow$	Process Calculus

Propositions as Sessions

propositions	*as*	types
proofs	*as*	programs
normalisation of proofs	*as*	evaluation of programs

propositions	*as*	session types
proofs	*as*	processes
cut elimination	*as*	communication

Lines of development

$$
\begin{tikzcd}
 & \text{Girard, 1987} \arrow[dl] \arrow[dr] & \\
\text{Abramsky, 1994} \arrow[d] & & \text{Honda, 1993} \arrow[d] \\
\text{Bellin and Scott, 1994} \arrow[d] & & \text{Honda, Vasconcelos, and Kubo, 1998} \arrow[dll] \arrow[d] \\
\text{Caires and Pfenning, 2010} \arrow[dr] & & \text{Gay and Vasconcelos, 2010} \arrow[dl] \\
 & \text{this paper} &
\end{tikzcd}
$$

The Twist

- Abramsky, 1994: Proofs as Processes

$$\frac{P \vdash \Gamma, y : A \qquad Q \vdash \Delta, z : B}{\nu y, z.\, x\langle y, z\rangle.(P \mid Q) \vdash \Gamma, \Delta, x : A \otimes B} \otimes$$

$$\frac{R \vdash \Theta, y : A, z : B}{x(y, z).R \vdash \Theta, x : A \parr B} \parr$$

- this paper: Propositions as Sessions

$$\frac{P \vdash \Gamma, y : A \qquad Q \vdash \Delta, x : B}{\nu y.\, x\langle y\rangle.(P \mid Q) \vdash \Gamma, \Delta, x : A \otimes B} \otimes$$

$$\frac{R \vdash \Theta, y : A, x : B}{x(y).R \vdash \Theta, x : A \parr B} \parr$$

A small change in notation

- With $\nu x.\, x\langle y\rangle$

$$\frac{P \vdash \Gamma, y : A \qquad Q \vdash \Delta, x : B}{\nu y.\, x\langle y\rangle.(P \mid Q) \vdash \Gamma, \Delta, x : A \otimes B} \otimes$$

- With $x[y]$

$$\frac{P \vdash \Gamma, y : A \qquad Q \vdash \Delta, x : B}{x[y].(P \mid Q) \vdash \Gamma, \Delta, x : A \otimes B} \otimes$$

ILL vs. CLL

- Caires and Pfenning, 2010: Intuitionistic Linear Logic

$$\frac{\Gamma;\ \Delta \vdash P :: y : A \qquad \Gamma;\ \Delta' \vdash Q :: x : B}{\Gamma;\ \Delta, \Delta' \vdash \nu y.\, x\langle y\rangle.(P \mid Q) :: x : A \otimes B} \otimes\text{-R}$$

$$\frac{\Gamma;\ \Delta \vdash P :: y : A \quad \Gamma;\ \Delta', x : B \vdash Q :: z : C}{\Gamma;\ \Delta, \Delta', x : A \multimap B \vdash \nu y.x\langle y\rangle.(P \mid Q) :: z : C} \multimap\text{-L}$$

$$\frac{\Gamma;\ \Delta, y : A \vdash R :: x : B}{\Gamma;\ \Delta \vdash x(y).R :: x : A \multimap B} \multimap\text{-R} \qquad \frac{\Gamma;\ \Delta, y : A, x : B \vdash R :: z : C}{\Gamma;\ \Delta, x : A \otimes B \vdash x(y).R :: z : C} \otimes\text{-L}$$

- this paper: Classical Linear Logic

$$\frac{P \vdash \Gamma, y : A \qquad Q \vdash \Delta, x : B}{x[y].(P \mid Q) \vdash \Gamma, \Delta, x : A \otimes B} \otimes \qquad \frac{R \vdash \Theta, y : A, x : B}{x(y).R \vdash \Theta, x : A \mathbin{⅋} B} ⅋$$

Axiom and Polymorphism

- Abramsky, 1994

$$\dfrac{}{x(z).w\langle z\rangle.0 \vdash w : X^{\perp},\; x : X}\;\mathsf{Ax}$$

- Caires and Pfenning, 2010

(no axiom)

- this paper (based on an idea from Caires and Pfenning, 2011)

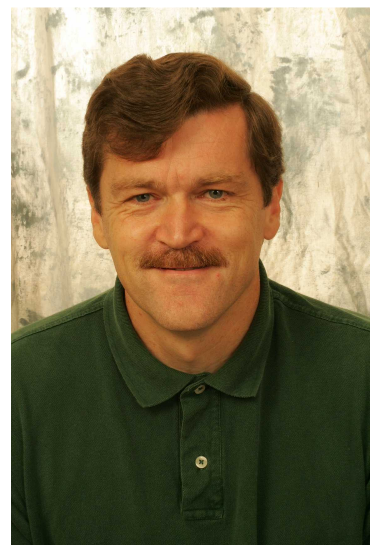
$$\dfrac{}{w{\leftrightarrow}x \vdash w : A^{\perp},\; x : A}\;\mathsf{Ax}$$

Part I

CP

Classical Processes

Caires-Pfenning



Cut Elimination

Theorem

(Subject Reduction)
If $P \vdash \Gamma$ and $P \Longrightarrow Q$ then $Q \vdash \Gamma$.

(Cut Elimination)
If $P \vdash \Gamma$ then there exists a Q
such that $P \Longrightarrow^* Q$ and Q is not a Cut.

Types

$A, B, C ::=$

X	type variable	$X^{\perp}$	dual of type variable
$A \otimes B$	output A then behave as B	$A \parr B$	input A then behave as B
$A \oplus B$	select from A or B	$A \mathbin{\&} B$	offer choice of A or B
$!A$	replicated input	$?A$	replicated output
$\exists X.B$	output a type	$\forall X.B$	input a type
1	unit for $\otimes$	$\perp$	unit for $\parr$
0	unit for $\oplus$	$\top$	unit for $\&$

Duals

$$
\begin{aligned}
(X)^\perp &= X^\perp & \qquad (X^\perp)^\perp &= X \\
(A \otimes B)^\perp &= A^\perp \parr B^\perp & \qquad (A \parr B)^\perp &= A^\perp \otimes B^\perp \\
(A \oplus B)^\perp &= A^\perp \mathbin{\&} B^\perp & \qquad (A \mathbin{\&} B)^\perp &= A^\perp \oplus B^\perp \\
(!A)^\perp &= ?A^\perp & \qquad (?A)^\perp &= !A^\perp \\
(\exists X.B)^\perp &= \forall X.B^\perp & \qquad (\forall X.B)^\perp &= \exists X.B^\perp \\
1^\perp &= \perp & \qquad \perp^\perp &= 1 \\
0^\perp &= \top & \qquad \top^\perp &= 0
\end{aligned}
$$

Processes

$P, Q, R ::=$

$x \leftrightarrow y$	link	$\nu x : A.(P \mid Q)$	parallel composition
$x[y].(P \mid Q)$	output	$x(y).P$	input
$x[\mathsf{inl}].P$	left selection	$x.\mathsf{case}(P, Q)$	choice
$x[\mathsf{inr}].P$	right selection		
$?x[y].P$	replicated output	$!x(y).P$	replicated input
$x[A].P$	send type	$x(X).P$	receive type
$x[\,].0$	empty output	$x(\,).P$	empty input
		$x.\mathsf{case}()$	empty choice

Forms $x(y).P$ and $!x(y).P$ behave like the same forms in π-calculus.

Forms $x[y].P$ and $?x[y].P$ behave like form $\nu y.\, x\langle y\rangle.P$ in π-calculus.

Structural rules

$$\frac{}{w{\leftrightarrow}x \vdash w : A^{\perp},\ x : A}\ \mathsf{Ax}$$

$$\frac{P \vdash \Gamma,\ x : A \qquad Q \vdash \Delta,\ x : A^{\perp}}{\nu x : A.(P \mid Q) \vdash \Gamma,\ \Delta}\ \mathsf{Cut}$$

(AxCut)

$$\frac{\dfrac{}{w{\leftrightarrow}x \vdash w : A^{\perp},\ x : A}\ \mathsf{Ax} \qquad P \vdash \Gamma,\ x : A^{\perp}}{\nu x.(w{\leftrightarrow}x \mid P) \vdash \Gamma,\ w : A^{\perp}}\ \mathsf{Cut} \quad \Longrightarrow \quad P\{w/x\} \vdash \Gamma,\ w : A^{\perp}$$

Structural rules—equivalences

(Swap)

$$\frac{P \vdash \Gamma, x : A \quad Q \vdash \Delta, x : A^{\perp}}{\nu x : A.(P \mid Q) \vdash \Gamma, \Delta}\ \textsf{Cut} \quad \equiv \quad \frac{Q \vdash \Delta, x : A^{\perp} \quad P \vdash \Gamma, x : A}{\nu x : A^{\perp}.(Q \mid P) \vdash \Gamma, \Delta}\ \textsf{Cut}$$

(Assoc)

$$\frac{\dfrac{P \vdash \Gamma, x : A \quad Q \vdash \Delta, x : A^{\perp}, y : B}{\nu x.(P \mid Q) \vdash \Gamma, \Delta, y : B}\ \textsf{Cut} \qquad R \vdash \Theta, y : B^{\perp}}{\nu y.(\nu x.(P \mid Q) \mid R) \vdash \Gamma, \Delta, \Theta}\ \textsf{Cut} \quad \equiv$$

$$\frac{P \vdash \Gamma, x : A \qquad \dfrac{Q \vdash \Delta, x : A^{\perp}, y : B \quad R \vdash \Theta, y : B^{\perp}}{\nu y.(Q \mid R) \vdash \Delta, \Theta, x : A^{\perp}}\ \textsf{Cut}}{\nu x.(P \mid \nu y.(Q \mid R)) \vdash \Gamma, \Delta, \Theta}\ \textsf{Cut}$$

Input and Output—Multiplicatives

$$\frac{P \vdash \Gamma, y : A \quad Q \vdash \Delta, x : B}{x[y].(P \mid Q) \vdash \Gamma, \Delta, x : A \otimes B} \otimes$$

$$\frac{R \vdash \Theta, y : A, x : B}{x(y).R \vdash \Theta, x : A \invamp B} \invamp$$

$(\beta_{\otimes \invamp})$

$$\dfrac{\dfrac{P \vdash \Gamma, y : A \quad Q \vdash \Delta, x : B}{x[y].(P \mid Q) \vdash \Gamma, \Delta, x : A \otimes B} \otimes \quad \dfrac{R \vdash \Theta, y : A^\perp, x : B^\perp}{x(y).R \vdash \Theta, x : A^\perp \invamp B^\perp} \invamp}{\nu x.(x[y].(P \mid Q) \mid x(y).R) \vdash \Gamma, \Delta, \Theta} \text{Cut} \quad \Longrightarrow$$

$$\dfrac{P \vdash \Gamma, y : A \quad \dfrac{Q \vdash \Delta, x : B \quad R \vdash \Theta, y : A^\perp, x : B^\perp}{\nu x.(Q \mid R) \vdash \Delta, \Theta, y : A^\perp} \text{Cut}}{\nu y.(P \mid \nu x.(Q \mid R)) \vdash \Gamma, \Delta, \Theta} \text{Cut}$$

Input and Output—identity, swap

$$\dfrac{\dfrac{}{y\leftrightarrow x \vdash y : A^\perp,\ x : A}\,\mathsf{Ax} \qquad \dfrac{}{w\leftrightarrow z \vdash w : B^\perp,\ z : B}\,\mathsf{Ax}}{\dfrac{z[x].(w\leftrightarrow z \mid y\leftrightarrow x) \vdash y : A^\perp,\ w : B^\perp,\ z : A \otimes B}{w(y).z[x].(w\leftrightarrow z \mid y\leftrightarrow x) \vdash w : A^\perp \parr B^\perp,\ z : A \otimes B}\,\parr}\,\otimes$$

$$\dfrac{\dfrac{}{w\leftrightarrow z \vdash w : B^\perp,\ z : B}\,\mathsf{Ax} \qquad \dfrac{}{y\leftrightarrow x \vdash y : A^\perp,\ x : A}\,\mathsf{Ax}}{\dfrac{x[z].(w\leftrightarrow z \mid y\leftrightarrow x) \vdash w : B^\perp,\ y : A^\perp,\ x : B \otimes A}{w(y).x[z].(w\leftrightarrow z \mid y\leftrightarrow x) \vdash w : A^\perp \parr B^\perp,\ x : B \otimes A}\,\parr}\,\otimes$$

Selection and Choice—Additives

$$\frac{P \vdash \Gamma,\ x : A}{x[\mathsf{inl}].P \vdash \Gamma,\ x : A \oplus B}\ \oplus_1 \qquad \frac{P \vdash \Gamma,\ x : B}{x[\mathsf{inr}].P \vdash \Gamma,\ x : A \oplus B}\ \oplus_2$$

$$\frac{Q \vdash \Delta,\ x : A \quad R \vdash \Delta,\ x : B}{x.\mathsf{case}(Q, R) \vdash \Delta,\ x : A \mathbin{\&} B}\ \&$$

$(\beta_{\oplus\&})$

$$\frac{\dfrac{P \vdash \Gamma,\ x : A}{x[\mathsf{inl}].P \vdash \Gamma,\ x : A \oplus B}\ \oplus_1 \quad \dfrac{Q \vdash \Delta,\ x : A^\perp \quad R \vdash \Delta,\ x : B^\perp}{x.\mathsf{case}(Q, R) \vdash \Delta,\ x : A^\perp \mathbin{\&} B^\perp}\ \&}{\nu x.(x[\mathsf{inl}].P \mid x.\mathsf{case}(Q, R)) \vdash \Gamma,\ \Delta}\ \mathsf{Cut} \quad \Longrightarrow$$

$$\frac{P \vdash \Gamma,\ x : A \quad Q \vdash \Delta,\ x : A^\perp}{\nu x.(P \mid Q) \vdash \Gamma,\ \Delta}\ \mathsf{Cut}$$

Servers and Clients—Exponentials

$$\frac{P \vdash ?\Gamma,\, y : A}{!x(y).P \vdash ?\Gamma,\, x : !A}\ !$$

$$\frac{Q \vdash \Delta,\, y : A}{?x[y].Q \vdash \Delta,\, x : ?A}\ ?$$

$(\beta_{!?})$

$$\frac{\dfrac{P \vdash ?\Gamma,\, y : A}{!x(y).P \vdash ?\Gamma,\, x : !A}\ ! \qquad \dfrac{Q \vdash \Delta,\, y : A^{\perp}}{?x[y].Q \vdash \Delta,\, x : ?A^{\perp}}\ ?}{\nu x.(!x(y).P \mid ?x[y].Q) \vdash ?\Gamma,\, \Delta}\ \mathsf{Cut} \qquad \Longrightarrow$$

$$\frac{P \vdash ?\Gamma,\, y : A \qquad Q \vdash \Delta,\, y : A^{\perp}}{\nu y.(P \mid Q) \vdash ?\Gamma,\, \Delta}\ \mathsf{Cut}$$

Weakening and Contraction

$$\dfrac{Q \vdash \Delta}{Q \vdash \Delta,\ x : ?A}\ \textsf{Weaken}$$

$$\dfrac{Q \vdash \Delta,\ x : ?A,\ x' : ?A}{Q\{x/x'\} \vdash \Delta,\ x : ?A}\ \textsf{Contract}$$

$(\beta_{!W})$

$$\dfrac{\dfrac{P \vdash ?\Gamma,\ y : A}{!x(y).P \vdash ?\Gamma,\ x : !A}\ ! \qquad \dfrac{Q \vdash \Delta}{Q \vdash \Delta,\ x : ?A^{\perp}}\ \textsf{Weaken}}{\nu x.(!x(y).P \mid Q) \vdash ?\Gamma,\ \Delta}\ \textsf{Cut} \qquad \Longrightarrow$$

$$\dfrac{Q \vdash \Delta}{\dfrac{}{Q \vdash ?\Gamma,\ \Delta}}\ \textsf{Weaken}$$

Weakening and Contraction, continued

$(\beta_{!C})$

$$\dfrac{\dfrac{P \vdash ?\Gamma,\, y : A}{!x(y).P \vdash ?\Gamma,\, x : !A}\;! \qquad \dfrac{Q \vdash \Delta,\, x : ?A,\, x' : ?A}{Q\{x/x'\} \vdash \Delta,\, x : ?A}\;\textsf{Contract}}{\nu x.(!x(y).P \mid Q\{x/x'\}) \vdash ?\Gamma,\, \Delta}\;\textsf{Cut} \qquad \Longrightarrow$$

$$\dfrac{\dfrac{\dfrac{P \vdash ?\Gamma,\, y : A}{!x(y).P \vdash ?\Gamma,\, x : !A}\;! \qquad \dfrac{\dfrac{P' \vdash ?\Gamma',\, y' : A}{!x'(y').P' \vdash ?\Gamma',\, x' : !A}\;! \qquad Q \vdash \Delta,\, x : ?A^{\perp},\, x' : ?A^{\perp}}{\nu x'.(!x'(y').P' \mid Q) \vdash ?\Gamma',\, \Delta,\, x : ?A^{\perp}}\;\textsf{Cut}}{\nu x.(!x(y).P \mid \nu x'.(!x'(y').P' \mid Q)) \vdash ?\Gamma,\, ?\Gamma',\, \Delta}\;\textsf{Cut}}{\nu x.(!x(y).P \mid \nu x'.(!x'(y).P \mid Q)) \vdash ?\Gamma,\, \Delta}\;\textsf{Contract}$$

Polymorphism—Quantifiers

$$\frac{P \vdash \Gamma,\ x : B\{A/X\}}{x[A].P \vdash \Gamma,\ x : \exists X.B}\ \exists$$

$$\frac{Q \vdash \Delta,\ x : B}{x(X).Q \vdash \Delta,\ x : \forall X.B}\ \forall \quad (X \notin \mathsf{fv}(\Delta))$$

$(\beta_{\exists\forall})$

$$\frac{\dfrac{P \vdash \Gamma,\ x : B\{A/X\}}{x[A].P \vdash \Gamma,\ x : \exists X.B}\ \exists \quad \dfrac{Q \vdash \Delta,\ x : B^{\perp}}{x(X).Q \vdash \Delta,\ x : \forall X.B^{\perp}}\ \forall}{\nu x.(x[A].P \mid x(X).Q) \vdash \Gamma,\ \Delta}\ \mathsf{Cut} \quad \Longrightarrow$$

$$\frac{P \vdash \Gamma,\ x : B\{A/X\} \quad Q\{A/X\} \vdash \Delta,\ x : B^{\perp}\{A/X\}}{\nu x.(P \mid Q\{A/X\}) \vdash \Gamma,\ \Delta}\ \mathsf{Cut}$$

Units

$$\frac{}{x[\,].0 \vdash x : 1}\ 1 \qquad \frac{P \vdash \Gamma}{x().P \vdash \Gamma,\ x : \bot}\ \bot$$

(no rule for 0) $$\frac{}{x.\mathsf{case}() \vdash \Gamma,\ x : \top}\ \top$$

$(\beta_{1\bot})$

$$\frac{\dfrac{}{x[\,].0 \vdash x : 1}\ 1 \quad \dfrac{P \vdash \Gamma}{x().P \vdash \Gamma,\ x : \bot}\ \bot}{\nu x.(x[\,].0 \mid x().P) \vdash \Gamma}\ \mathsf{Cut} \quad \Longrightarrow \quad P \vdash \Gamma$$

$(\beta_{0\top})$

(no rule for 0 with $\top$)

Commuting Conversions

$(\kappa_{\otimes 1})$

$$\dfrac{\dfrac{P \vdash \Gamma, y : A, z : C \quad Q \vdash \Delta, x : B}{x[y].(P \mid Q) \vdash \Gamma, \Delta, x : A \otimes B}\otimes \qquad R \vdash \Theta, z : C^\perp}{\nu z.(x[y].(P \mid Q) \mid R) \vdash \Gamma, \Delta, \Theta, x : A \otimes B}\,\mathsf{Cut} \quad \Longrightarrow$$

$$\dfrac{\dfrac{P \vdash \Gamma, y : A, z : C \quad R \vdash \Theta, z : C^\perp}{\nu z.(P \mid R) \vdash \Gamma, \Theta, y : A}\,\mathsf{Cut} \qquad Q \vdash \Delta, x : B}{x[y].(\nu z.(P \mid R) \mid Q) \vdash \Gamma, \Delta, \Theta, x : A \otimes B}\otimes$$

$(\kappa_{\otimes 2})$

$$\dfrac{\dfrac{P \vdash \Gamma, y : A \quad Q \vdash \Delta, x : B, z : C}{x[y].(P \mid Q) \vdash \Gamma, \Delta, x : A \otimes B}\otimes \qquad R \vdash \Theta, z : C^\perp}{\nu z.(x[y].(P \mid Q) \mid R) \vdash \Gamma, \Delta, \Theta, x : A \otimes B}\,\mathsf{Cut} \quad \Longrightarrow$$

$$\dfrac{P \vdash \Gamma, y : A \qquad \dfrac{Q \vdash \Delta, x : B, z : C \quad R \vdash \Theta, z : C^\perp}{\nu z.(Q \mid R) \vdash \Delta, \Theta, x : B}\,\mathsf{Cut}}{x[y].(P \mid \nu z.(Q \mid R)) \vdash \Gamma, \Delta, \Theta, x : A \otimes B}\otimes$$

Commuting Conversions

$$
\begin{array}{llcll}
(\kappa_{\otimes 1}) & \nu z.(x[y].(P \mid Q) \mid R) & \Longrightarrow & x[y].(\nu z.(P \mid R) \mid Q), & \text{if } z \in \mathsf{fn}(P) \\
(\kappa_{\otimes 2}) & \nu z.(x[y].(P \mid Q) \mid R) & \Longrightarrow & x[y].(P \mid \nu z.(Q \mid R)), & \text{if } z \in \mathsf{fn}(Q) \\
(\kappa_{\otimes}) & \nu z.(x(y).P \mid Q) & \Longrightarrow & x(y).\nu z.(P \mid Q) & \\
(\kappa_{\oplus}) & \nu z.(x[\mathsf{inl}].P \mid Q) & \Longrightarrow & x[\mathsf{inl}].\nu z.(P \mid Q) & \\
(\kappa_{\&}) & \nu z.(x.\mathsf{case}(P, Q) \mid R) & \Longrightarrow & x.\mathsf{case}(\nu z.(P \mid R), \nu z.(Q \mid R)) & \\
(\kappa_{!}) & \nu z.(!x(y).P \mid Q) & \Longrightarrow & !x(y).\nu z.(P \mid Q) & \\
(\kappa_{?}) & \nu z.(?x[y].P \mid Q) & \Longrightarrow & ?x[y].\nu z.(P \mid Q) & \\
(\kappa_{\exists}) & \nu z.(x[A].P \mid Q) & \Longrightarrow & x[A].\nu z.(P \mid Q) & \\
(\kappa_{\forall}) & \nu z.(x(X).P \mid Q) & \Longrightarrow & x(X).\nu z.(P \mid Q) & \\
(\kappa_{\bot}) & \nu z.(x().P \mid Q) & \Longrightarrow & x().\nu z.(P \mid Q) & \\
(\kappa_{\top}) & \nu z.(x.\mathsf{case}() \mid Q) & \Longrightarrow & x.\mathsf{case}() &
\end{array}
$$

No congruence!

If our goal was to eliminate all cuts, we would need to introduce congruence rules, such as

$$\frac{P \Longrightarrow Q}{x(y).P \Longrightarrow x(y).Q}$$

and similarly for each operator. Such rules do not correspond well to our notion of computation on processes, so we omit them; this is analogous to the usual practice of not permitting reduction under lambda.

Cut Elimination

Theorem

(Subject Reduction)
If $P \vdash \Gamma$ and $P \Longrightarrow Q$ then $Q \vdash \Gamma$.

(Cut Elimination)
If $P \vdash \Gamma$ then there exists a Q
such that $P \Longrightarrow^* Q$ and Q is not a Cut.

Part II

GV

Good Variation

Gay-Vasconcelos

Type Preservation

Theorem

(Translation preserves types)

If $\Phi \vdash M : T$

then $[\![M]\!]z \vdash [\![\Phi]\!]^{\perp},\ z : [\![T]\!]$.

Session Types

$S ::=$

$!T.S$	output value of type T then behave as S
$?T.S$	input value of type T then behave as S
$\oplus\{l_i : S_i\}_{i\in I}$	select from behaviours S_i with label l_i
$\&\{l_i : S_i\}_{i\in I}$	offer choice of behaviours S_i with label l_i
$\mathsf{end}_!$	terminator, convenient for use with output
$\mathsf{end}_?$	terminator, convenient for use with input

Each session S has a dual $\overline{S}$:

$$
\begin{array}{rclcrcl}
\overline{!T.S} & = & ?T.\overline{S} & \quad & \overline{?T.S.} & = & !T.\overline{S} \\
\overline{\oplus(l_i : S_i)_{i\in I}} & = & \&(l_i : \overline{S}_i)_{i\in I} & & \overline{\&(l_i : S_i)_{i\in I}} & = & \oplus(l_i : \overline{S}_i)_{i\in I} \\
\overline{\mathsf{end}_!} & = & \mathsf{end}_? & & \overline{\mathsf{end}_?} & = & \mathsf{end}_!
\end{array}
$$

Types

$$T, U, V ::=$$

S	session (linear)
$T \otimes U$	tensor product (linear)
$T \multimap U$	function (linear)
$T \to U$	function (unlimited)
Unit	unit (unlimited)

Each type is classified as linear or unlimited:

$$\mathsf{lin}(S) \quad \mathsf{lin}(T \otimes U) \quad \mathsf{lin}(T \multimap U)$$

$$\mathsf{un}(T \to U) \quad \mathsf{un}(\mathsf{Unit})$$

Terms

$L, M, N ::=$

x	identifier
unit	unit constant
$\lambda x.\, N$	function abstraction
$L\ M$	function application
(M, N)	pair construction
$\mathsf{let}\ (x, y) = M\ \mathsf{in}\ N$	pair deconstruction
$\mathsf{send}\ M\ N$	send value M on channel N
$\mathsf{receive}\ M$	receive from channel M
$\mathsf{select}\ l\ M$	select label l on channel M
$\mathsf{case}\ M\ \mathsf{of}\ \{l_i : x.N_i\}_{i \in I}$	offer choice on channel M
$\mathsf{with}\ x\ \mathsf{connect}\ M\ \mathsf{to}\ N$	connect M to N by channel x
$\mathsf{terminate}\ M$	terminate input

Functions and Pairs

$$\frac{}{x : T \vdash x : T}\ \textsf{Id} \qquad \frac{}{\vdash \textsf{unit} : \textsf{Unit}}\ \textsf{Unit}$$

$$\frac{\Phi \vdash N : U \quad \textsf{un}(T)}{\Phi,\ x : T \vdash N : U}\ \textsf{Weaken} \qquad \frac{\Phi,\ x : T,\ x' : T \vdash N : U \quad \textsf{un}(T)}{\Phi,\ x : T \vdash N\{x/x'\} : U}\ \textsf{Contract}$$

$$\frac{\Phi,\ x : T \vdash N : U}{\Phi \vdash \lambda x.\ N : T \multimap U}\ \multimap\textsf{-I} \qquad \frac{\Phi \vdash L : T \multimap U \quad \Psi \vdash M : T}{\Phi,\ \Psi \vdash L\ M : U}\ \multimap\textsf{-E}$$

$$\frac{\Phi \vdash L : T \multimap U \quad \textsf{un}(\Phi)}{\Phi \vdash L : T \to U}\ \to\textsf{-I} \qquad \frac{\Phi \vdash L : T \to U}{\Phi \vdash L : T \multimap U}\ \to\textsf{-E}$$

$$\frac{\Phi \vdash M : T \quad \Psi \vdash N : U}{\Phi,\ \Psi \vdash (M, N) : T \otimes U}\ \otimes\textsf{-I} \qquad \frac{\Phi \vdash M : T \otimes U \quad \Psi,\ x : T,\ y : U \vdash N : V}{\Phi,\ \Psi \vdash \textsf{let}\ (x, y) = M\ \textsf{in}\ N : V}\ \otimes\textsf{-E}$$

Communication

$$\frac{\Phi \vdash M : T \quad \Psi \vdash N : !T.S}{\Phi, \Psi \vdash \textsf{send}\ M\ N : S}\ \textsf{Send} \qquad \frac{\Phi \vdash M : ?T.S}{\Phi \vdash \textsf{receive}\ M : T \otimes S}\ \textsf{Receive}$$

$$\frac{\Phi \vdash M : \oplus\{l_i : S_i\}_{i\in I}}{\Phi \vdash \textsf{select}\ l_j\ M : S_j}\ \textsf{Select}$$

$$\frac{\Phi \vdash M : \&\{l_i : S_i\}_{i\in I} \quad (\Psi, x : S_i \vdash N_i : T)_{i\in I}}{\Phi, \Psi \vdash \textsf{case}\ M\ \textsf{of}\ \{l_i : x.N_i\}_{i\in I} : T}\ \textsf{Case}$$

$$\frac{\Phi, x : S \vdash M : \textsf{end}_! \quad \Psi, x : \overline{S} \vdash N : T}{\Phi, \Psi \vdash \textsf{with}\ x\ \textsf{connect}\ M\ \textsf{to}\ N : T}\ \textsf{Connect}$$

$$\frac{\Phi \vdash M : T \otimes \textsf{end}_?}{\Phi \vdash \textsf{terminate}\ M : T}\ \textsf{Terminate}$$

Translation of Sessions

$$
\begin{aligned}
[\![!T.S]\!] &= [\![T]\!]^{\perp} \parr [\![S]\!] \\
[\![?T.S]\!] &= [\![T]\!] \otimes [\![S]\!] \\
[\![\oplus\{l_i : S_i\}_{i\in I}]\!] &= [\![S_1]\!] \,\&\, \cdots \,\&\, [\![S_n]\!], \quad I = \{1, \ldots, n\} \\
[\![\&\{l_i : S_i\}_{i\in I}]\!] &= [\![S_1]\!] \oplus \cdots \oplus [\![S_n]\!], \quad I = \{1, \ldots, n\} \\
[\![\mathsf{end}_!]\!] &= \perp \\
[\![\mathsf{end}_?]\!] &= 1
\end{aligned}
$$

Translation preserves duality:

$$
[\![\overline{S}]\!] \quad = \quad [\![S]\!]^{\perp}
$$

Translation of Types

$$
\begin{aligned}
[\![T \multimap U]\!] &= [\![T]\!]^{\perp} \parr [\![U]\!] \\
[\![T \rightarrow U]\!] &= !([\![T]\!]^{\perp} \parr [\![U]\!]) \\
[\![T \otimes U]\!] &= [\![T]\!] \otimes [\![U]\!] \\
[\![\mathsf{Unit}]\!] &= !\top
\end{aligned}
$$

An unlimited type translates to a type constructed with !:

$$\text{If } \ \mathsf{un}(T) \ \text{ then } \ [\![T]\!] = !A, \ \text{ for some } A.$$

Translation of Linear Functions

$$\left[\!\!\left[\frac{\Phi,\, x : T \vdash N : U}{\Phi \vdash \lambda x.\, N : T \multimap U} \multimap\text{-I} \right]\!\!\right] z \quad =$$

$$\frac{[\![N]\!]z \vdash [\![\Phi]\!]^\perp,\, x : [\![T]\!]^\perp,\, z : [\![U]\!]}{z(x).[\![N]\!]z \vdash [\![\Phi]\!]^\perp,\, z : [\![T]\!]^\perp \parr [\![U]\!]} \parr$$

$$\left[\!\!\left[\frac{\Phi \vdash L : T \multimap U \qquad \Psi \vdash M : T}{\Phi,\, \Psi \vdash L\, M : U} \multimap\text{-E} \right]\!\!\right] z \quad =$$

$$\frac{[\![L]\!]y \vdash [\![\Phi]\!]^\perp,\, y : [\![T]\!]^\perp \parr [\![U]\!] \qquad \dfrac{[\![M]\!]x \vdash [\![\Psi]\!]^\perp,\, x : [\![T]\!] \qquad y{\leftrightarrow}z \vdash y : [\![U]\!]^\perp,\, z : [\![U]\!]}{y[x].([\![M]\!]x \mid y{\leftrightarrow}z) \vdash [\![\Psi]\!]^\perp,\, y : [\![T]\!] \otimes [\![U]\!]^\perp,\, z : [\![U]\!]} \otimes}{\nu y.([\![L]\!]y \mid y[x].([\![M]\!]x \mid y{\leftrightarrow}z)) \vdash [\![\Phi]\!]^\perp,\, [\![\Psi]\!]^\perp,\, z : [\![U]\!]} \text{Cut}$$

Translation of Unlimited Functions

$$\left[\!\!\left[\frac{\Phi \vdash L : T \multimap U \quad \mathsf{un}(\Phi)}{\Phi \vdash L : T \to U} \to\text{-I} \right]\!\!\right] z \quad = $$

$$\frac{[\![L]\!] y \vdash [\![\Phi]\!]^{\perp},\ y : [\![T \multimap U]\!]}{!z(y).[\![L]\!] y \vdash [\![\Phi]\!]^{\perp},\ z : ![\![T \multimap U]\!]}\ !$$

$$\left[\!\!\left[\frac{\Phi \vdash L : T \to U}{\Phi \vdash L : T \multimap U} \to\text{-E} \right]\!\!\right] z \quad = $$

$$\frac{[\![L]\!] y \vdash [\![\Phi]\!]^{\perp},\ y : ![\![T \multimap U]\!] \qquad \dfrac{\dfrac{}{x{\leftrightarrow}z \vdash x : [\![T \multimap U]\!]^{\perp},\ z : [\![T \multimap U]\!]}\ \mathsf{Ax}}{?y[x].x{\leftrightarrow}z \vdash y : ?[\![T \multimap U]\!]^{\perp},\ z : [\![T \multimap U]\!]}\ ?}{\nu y.([\![L]\!] y \mid ?y[x].x{\leftrightarrow}z) \vdash [\![\Phi]\!]^{\perp},\ z : [\![T \multimap U]\!]}\ \mathsf{Cut}$$

Translation of Send and Receive

$$\left[\!\left[\frac{\Phi \vdash M : T \quad \Psi \vdash N : !T.S}{\Phi, \Psi \vdash \mathsf{send}\ M\ N : S}\ \mathsf{Send} \right]\!\right] z \quad =$$

$$\dfrac{\dfrac{[\![M]\!]y \vdash [\![\Phi]\!]^{\perp}, y : [\![T]\!] \quad x{\leftrightarrow}z \vdash x : [\![S]\!]^{\perp}, z : [\![S]\!]}{x[y].([\![M]\!]y \mid x{\leftrightarrow}z) \vdash [\![\Phi]\!]^{\perp}, x : [\![T]\!] \otimes [\![S]\!]^{\perp}}\ \otimes \qquad [\![N]\!]x \vdash [\![\Psi]\!]^{\perp}, x : [\![T]\!]^{\perp} \parr [\![S]\!]}{\nu x.(x[y].([\![M]\!]y \mid x{\leftrightarrow}z) \mid [\![N]\!]x) \vdash [\![\Phi]\!]^{\perp}, [\![\Psi]\!]^{\perp}, z : [\![S]\!]}\ \mathsf{Cut}$$

$$\left[\!\left[\frac{\Phi \vdash M : ?T.S}{\Phi \vdash \mathsf{receive}\ M : T \otimes S}\ \mathsf{Receive} \right]\!\right] z \quad =$$

$$[\![M]\!]z \vdash [\![\Phi]\!], z : [\![T]\!] \otimes [\![S]\!]$$

Translation of Connect and Terminate

$$\left[\!\!\left[\frac{\Phi,\, x : S \vdash M : \mathsf{end}_! \qquad \Psi,\, x : \overline{S} \vdash N : T}{\Phi,\, \Psi \vdash \mathsf{with}\ x\ \mathsf{connect}\ M\ \mathsf{to}\ N : T}\ \textsf{Connect}\right]\!\!\right] z \quad =$$

$$\dfrac{\dfrac{[\![M]\!]y \vdash [\![\Phi]\!]^\perp,\, x : [\![S]\!]^\perp,\, y : \bot \qquad \dfrac{}{y[].0 \vdash y : \mathbf{1}}\ \textsf{1}}{\nu y.([\![M]\!]y \mid y[].0) \vdash [\![\Phi]\!]^\perp,\, x : [\![S]\!]^\perp}\ \textsf{Cut} \qquad [\![N]\!]z \vdash [\![\Psi]\!]^\perp,\, x : [\![S]\!],\, z : [\![T]\!]}{\nu x.(\nu y.([\![M]\!]y \mid y[].0) \mid [\![N]\!]z) \vdash [\![\Phi]\!]^\perp,\, [\![\Psi]\!]^\perp,\, z : [\![T]\!]}$$

$$\left[\!\!\left[\frac{\Phi \vdash M : T \otimes \mathsf{end}_?}{\Phi \vdash \mathsf{terminate}\ M : T}\ \textsf{Terminate}\right]\!\!\right] z \quad =$$

$$\dfrac{[\![M]\!]y \vdash [\![\Phi]\!]^\perp,\, y : [\![T]\!] \otimes \mathbf{1} \qquad \dfrac{\dfrac{z \leftrightarrow y \vdash z : [\![T]\!],\, y : [\![T]\!]^\perp}{x().z \leftrightarrow y \vdash z : [\![T]\!],\, y : [\![T]\!]^\perp,\, x : \bot}\ \bot}{y(x).x().z \leftrightarrow y \vdash z : [\![T]\!],\, y : [\![T]\!]^\perp \mathbin{⅋} \bot}\ ⅋}{\nu y.([\![M]\!]y \mid y(x).x().z \leftrightarrow y) \vdash [\![\Phi]\!]^\perp,\, z : [\![T]\!]}\ \textsf{Cut}$$

Type Preservation

Theorem

(Translation preserves types)

If $\Phi \vdash M : T$

then $[\![M]\!]z \vdash [\![\Phi]\!]^{\perp}, z : [\![T]\!]$.

Part III

Conclusions and future work

Paradoxical combinator

$$X = X \supset A$$

$$\dfrac{\dfrac{\dfrac{[x : X \supset A]^x \quad [x : X]^x}{x\; x : A}\; \supset\text{-E}}{\lambda x.\, x\; x : X \supset A}\; \supset\text{-I}^x \qquad \dfrac{\dfrac{[x : X \supset A]^x \quad [x : X]^x}{x\; x : A}\; \supset\text{-E}}{\lambda x.\, x\; x : X}\; \supset\text{-I}^x}{(\lambda x.\, x\; x)\; (\lambda x.\, x\; x) : A}\; \supset\text{-E}$$

Fixpoint combinator

$$X = X \supset A$$

$$\dfrac{\dfrac{[f : A \supset A]^f \qquad \dfrac{[x : X \supset A]^x \quad [x : X]^x}{x\,x : A}\,\supset\text{-E}}{\dfrac{f\ (x\,x) : A}{\lambda x.\, f\ (x\,x) : X \supset A}\,\supset\text{-I}^x}\,\supset\text{-E} \qquad \dfrac{\dfrac{[f : A \supset A]^f \qquad \dfrac{[x : X \supset A]^x \quad [x : X]^x}{x\,x : A}\,\supset\text{-E}}{f\ (x\,x) : A}\,\supset\text{-E}}{\lambda x.\, f\ (x\,x) : X \supset A}\,\supset\text{-I}^x}{\dfrac{(\lambda x.\, f\ (x\,x))\ (\lambda x.\, f\ (x\,x)) : A}{\lambda f.\, (\lambda x.\, f\ (x\,x))\ (\lambda x.\, f\ (x\,x)) : (A \supset A) \supset A}\,\supset\text{-I}^f}\,\supset\text{-E}$$

Restoring the full power of π-calculus

- Mix rule, Girard (1987)

$$\frac{P \vdash \Gamma \quad Q \vdash \Delta}{P \mid Q \vdash \Gamma, \Delta}\ \textsf{Mix}$$

- Binary Cut rule, Abramsky, Gay, and Nagarajan (1996)

$$\frac{P \vdash \Gamma, x : A, y : B \quad Q \vdash \Delta, x : A^{\perp}, y : B^{\perp}}{\nu x : A, y : B.(P \mid Q) \vdash \Gamma, \Delta}\ \textsf{BiCut}$$