Errata:
Keeping partners together: Algorithmic results for the Hospitals / Residents problem with Couples

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Published in *Journal of Combinatorial Optimization*, volume 19, pages 279-303, 2010.

Theorem 3.8 and Corollary 3.9 in [2] are stated as follows:

Theorem 3.8. *The problem of determining whether an HRS instance admits a stable matching is NP-complete, even if the size of each resident and the capacity of each hospital is at most 2, and the lengths of the residents' and hospitals' preference lists are at most 3 (these conditions holding simultaneously).*

Corollary 3.9. *The problem of determining whether an HRCC instance admits a stable matching is NP-complete, even if the individual preference list of each resident and the joint preference list of each couple has at most 3 entries, and the capacity of each hospital is at most 2 (these conditions holding simultaneously).*

However in the reduction given in the proof of Theorem 3.8 in [2], some preference lists may in fact be of length 4 (namely those of residents of the form r_s). A similar remark holds for Corollary 3.9 (i.e., some couples' lists may contain as many as 4 pairs). In this note we present a revised proof of Theorem 3.8, which in turn establishes Corollary 3.9. In what follows we assume the notation and terminology used in [2].

Proof of Theorem 3.8. We reduce from a a restricted version of SAT. Let (2,2)-E3-SAT denote the problem of deciding, given a Boolean formula B in CNF in which each clause contains exactly 3 literals and, for each variable v_j, each of literals v_j and $\bar{v}_j$ appears exactly twice in B, whether B is satisfiable. Berman et al. [1] showed that (2,2)-E3-SAT is NP-complete.

Hence let B be an instance of (2,2)-E3-SAT. Let $V = \{v_1, v_2, \ldots, v_n\}$ and $C = \{c_1, c_2, \ldots, c_m\}$ be the set of variables and clauses respectively in B. Let us construct an instance of HRS in the following way.

For each variable v_j there are 6 residents $r_j^1, r_j^2, \ldots, r_j^6$, 4 residents $x_j^1, x_j^2, y_j^1, y_j^2$, 12 residents $q_{j,1}^k, q_{j,2}^k, q_{j,3}^k$ $(1 \leq k \leq 4)$, 6 hospitals $h_j^1, h_j^2, h_j^3, h_j^4, h_j^T, h_j^F$ and 12 hospitals

resident	size	preferences	hospital	capacity	preferences
r_j^1	2	$h_j^1 \ h_j^3$	h_j^1	2	$r_j^4 \ r_j^1 \ r_j^3$
r_j^2	2	$h_j^2 \ h_j^4$	h_j^2	2	$r_j^3 \ r_j^2 \ r_j^4$
r_j^3	1	$h_j^1 \ h_j^2$	h_j^3	2	$r_j^1 \ r_j^5$
r_j^4	1	$h_j^2 \ h_j^1$	h_j^4	2	$r_j^2 \ r_j^6$
r_j^5	2	$h_j^3 \ h_j^T$	h_j^T	2	$r_j^5 \ x_j^1 \ x_j^2$
r_j^6	2	$h_j^4 \ h_j^F$	h_j^F	2	$r_j^6 \ y_j^1 \ y_j^2$
x_j^1	1	$h_j^T \ z(x_j^1) \ p_{j,3}^1$	z_i	2	$v_i^1 \ v_i^2 \ v_i^3$
x_j^2	1	$h_j^T \ z(x_j^2) \ p_{j,3}^2$	$p_{j,1}^k$	2	$q_{j,1}^k \ q_{j,3}^k \ q_{j,2}^k$
y_j^1	1	$h_j^F \ z(y_j^1) \ p_{j,3}^3$	$p_{j,2}^k$	1	$q_{j,2}^k \ q_{j,1}^k$
y_j^2	1	$h_j^F \ z(y_j^2) \ p_{j,3}^4$	$p_{j,3}^k$	1	$v(p_{j,3}^k) \ q_{j,3}^k$
$q_{j,1}^k$	1	$p_{j,2}^k \ p_{j,1}^k$			
$q_{j,2}^k$	1	$p_{j,1}^k \ p_{j,2}^k$			
$q_{j,3}^k$	2	$p_{j,3}^k \ p_{j,1}^k$			

Figure 1: The constructed instance of HRS

$p_{j,1}^k, p_{j,2}^k, p_{j,3}^k$ $(1 \leq k \leq 4)$. For each clause c_i there is one hospital z_i. Residents x_j^1 and x_j^2 correspond to the first and second occurrence of literal v_j, whilst residents y_j^1 and y_j^2 correspond to the first and second occurrence of literal $\bar{v}_j$, respectively.

The characteristics of agents and their preferences are given in Figure 1. Here, the subscripts and superscripts involving i, j and k range over the following intervals: $1 \leq i \leq m$, $1 \leq j \leq n$ and $1 \leq k \leq 4$. In the preference list of hospital z_i, the symbol v_i^s means the x- or y-resident that corresponds to the literal that appears in position s of clause c_i. Conversely, in the preference list of x- or y-residents the symbol $z(.)$ denotes the z-hospital corresponding to the clause containing the corresponding literal. Also, in the preference list of $p_{j,3}^k$, the symbol $v(p_{j,3}^k)$ denotes x_j^k if $1 \leq k \leq 2$ and denotes y_j^{k-2} if $3 \leq k \leq 4$.

For each j, $1 \leq j \leq n$, let us denote

$$T_j = \{(x_j^1, h_j^T), (x_j^2, h_j^T), (r_j^6, h_j^F)\}, \qquad F_j = \{(y_j^1, h_j^F), (y_j^2, h_j^F), (r_j^5, h_j^T)\}.$$

For brevity, hospitals h_j^T and h_j^F will be called *decisive hospitals.*

Now, let f be a satisfying truth assignment of B. Define a matching M in I as follows. For each variable $v_j \in V$, if v_j is true under f, put the pairs T_j into M and if v_j is false under f put the pairs F_j into M. In the former case add the pairs

$$(y_j^1, z(y_j^1)), (y_j^2, z(y_j^2)), (r_j^1, h_j^1), (r_j^2, h_j^4), (r_j^3, h_j^2), (r_j^4, h_j^2), (r_j^5, h_j^3),$$

and in the latter case add the pairs

$$(x_j^1, z(x_j^1)), (x_j^2, z(x_j^2)), (r_j^1, h_j^3), (r_j^2, h_j^2), (r_j^3, h_j^1), (r_j^4, h_j^1), (r_j^6, h_j^4).$$

Notice that as each clause $c_i \in C$ contains at most two false literals, hospital z_i has enough capacity for accepting all the allocated residents. Finally, add the following pairs for each j $(1 \leq j \leq n)$ and k $(1 \leq k \leq 4)$:

$$(q_{j,1}^k, p_{j,2}^k), (q_{j,2}^k, p_{j,1}^k), (q_{j,3}^k, p_{j,3}^k).$$

It is obvious that the defined matching is feasible; it remains to prove that it is stable. We show this by considering each type of residents corresponding to variable v_j in turn. Firstly we remark that residents $q^k_{j,1}, q^k_{j,2}, q^k_{j,3}$ each have their first choice hospital ($1 \leq k \leq 4$) so cannot be involved in a blocking pair. Now suppose that v_j is true under f. Then:

- residents x^1_j, x^2_j, r^1_j, r^4_j and r^5_j have their most-preferred hospitals, so are not blocking.
- residents y^1_j and y^2_j prefer hospital h^F_j, but this hospital is fully occupied by r^6_j, whom it prefers.
- resident r^2_j prefers hospital h^2_j, but this hospital is full and does not prefer r^2_j to a set of applicants of size at least 2.
- resident r^3_j prefers hospital h^1_j, but this hospital is fully occupied by r^1_j, whom it prefers.
- resident r^6_j prefers hospital h^4_j, but this hospital is fully occupied by r^2_j, whom it prefers.

The case of a false variable can be proved similarly.

For the converse implication let us first prove two claims.

Claim 1. *Each stable matching M contains for each j either all the pairs in T_j or all the pairs in F_j.*

Proof. Let M be a stable matching. Fix $j \in \{1, 2, \ldots, n\}$. Notice first that both hospitals h^T_j and h^F_j must be full, otherwise either h^T_j will form a blocking pair with at least one of x^1_j and x^2_j, or h^F_j will form a blocking pair with at least one of y^1_j and y^2_j. Further, let us distinguish the following cases.

- $\{(r^5_j, h^T_j), (r^6_j, h^F_j)\} \subseteq M$. Then, as there are no blocking pairs, $\{(r^1_j, h^3_j), (r^2_j, h^4_j)\} \subseteq M$, which further implies $\{(r^3_j, h^2_j), (r^4_j, h^1_j)\} \subseteq M$. This, however means that (r^3_j, h^1_j) and (r^4_j, h^2_j) are blocking pairs for M, a contradiction.
- $\{(x^1_j, h^T_j), (x^2_j, h^T_j), (y^1_j, h^F_j), (y^2_j, h^F_j)\} \subseteq M$. Now, to avoid blocking pairs, $\{(r^5_j, h^3_j), (r^6_j, h^4_j)\} \subseteq M$, which further implies $\{(r^1_j, h^1_j), (r^2_j, h^2_j)\} \subseteq M$. Then there are blocking pairs (r^3_j, h^2_j) and (r^4_j, h^1_j), again a contradiction. □

Claim 2. *In each stable matching M every resident in the set $\{x^1_j, x^2_j, y^1_j, y^2_j : 1 \leq j \leq n\}$ is matched to her first- or second-choice hospital.*

Proof. For some j ($1 \leq j \leq n$), consider resident x^1_j (the argument for x^2_j, y^1_j, y^2_j is similar). Suppose firstly that x^1_j is unmatched in M. Then $(x^1_j, p^1_{j,3})$ blocks M, a contradiction. Now suppose that $(x^1_j, p^1_{j,3}) \in M$. If $(q^1_{j,3}, p^1_{j,1}) \in M$ then $(q^1_{j,1}, p^1_{j,2}) \in M$, for otherwise $(q^1_{j,1}, p^1_{j,1})$ blocks M. But then $(q^1_{j,2}, p^1_{j,2})$ blocks M, a contradiction. Thus $q^1_{j,3}$ is unmatched in M. Then $(q^1_{j,2}, p^1_{j,1}) \in M'$, for otherwise $(q^1_{j,2}, p^1_{j,1})$ blocks M. Also $(q^1_{j,1}, p^1_{j,2}) \in M'$, for otherwise $(q^1_{j,1}, p^1_{j,2})$ blocks M. Hence $(q^1_{j,3}, p^1_{j,1})$ blocks M, a contradiction. □

Conversely, suppose that M is a stable matching in I. We form a truth assignment f in B as follows. Let j ($1 \leq j \leq n$) be given. If $T_j \subseteq M$, set $f(v_j) = T$, otherwise set $f(v_j) = F$. Now let $v_j \in V$ and suppose that $f(v_j) = T$. Then by Claim 2, each of $y_{j,1}$ and $y_{j,2}$ is matched to her second choice hospital. Now suppose that $f(v_j) = F$. Then by Claims 1 and 2, each of $x_{j,1}$ and $x_{j,2}$ is matched to her second choice hospital. Now let $c_i \in C$ and suppose that all literals in c_i are false. By the preceding remarks about $x_{j,1}, x_{j,2}, y_{j,1}$ and $y_{j,2}$ we deduce that z_i is over-subscribed, a contradiction. Thus f is a satisfying truth assignment. □

Corollary 3.9 then follows immediately by Theorem 3.8 and by Lemma 2.1 in [2].

References

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