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Abstract

Popular matchings have recently been a subject of study in the context of the so-called *House Allocation Problem*, where the objective is to match applicants to houses over which the applicants have preferences. A matching M is called *popular* if there is no other matching M' with the property that more applicants prefer their allocation in M' to their allocation in M. In this paper we study popular matchings in the context of the *Roommates Problem*, including its special (bipartite) case, the *Marriage Problem*. We investigate the relationship between popularity and stability, and describe efficient algorithms to test a matching for popularity in these settings. We also show that, when ties are permitted in the preferences, it is NP-hard to determine whether a popular matching exists in both the Roommates and Marriage cases.

Keywords: popular matchings; stable matchings; Marriage problem; Roommates problem; computational complexity

1 Introduction

1.1 Background

Stable matching problems have a long history, dating back to the seminal paper of Gale and Shapley [10], and these problems continue as an area of active research among computer scientists, mathematicians and economists [14, 27]. An instance of the classical Stable Marriage problem (SM) involves sets of n men and n women, and each person has a strict order of preference (their *preference list*) over all of the members of the opposite sex. A *stable* matching M is a set of n disjoint man-woman pairs such that no man m and woman w who do not form a pair prefer each other to their partners in M. The Stable Roommates problem (SR) is the generalisation of SM to the non-bipartite case, where each person has a strict order of preference over all of the others.

Gale and Shapley [10] showed that every instance of SM admits a stable matching, and such a matching can be found in $O(n^2)$ time, whereas, by contrast, some SR instances admit no stable matching. Irving [16] gave an $O(n^2)$ time algorithm to find a stable matching in an SR instance, when one exists.

A wide range of extensions of these fundamental problems have been studied. For instance, the existence results and efficient algorithms extend to the case where preference lists are *incomplete*, i.e., when participants can declare some of the others to be unacceptable as partners. In this case, both the Gale-Shapley algorithm and Irving's algorithm can

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be adapted to run in $O(m)$ time, where m is the sum of the lengths of the preference lists [14]. Furthermore, in this case, it is known that all stable matchings have the same size and match exactly the same people [11, 14]. If, in addition, *ties* are permitted in an individual's preferences, then the situation becomes more complex. Here, a stable matching always exists, but different stable matchings can have different sizes, and it is NP-hard to find a stable matching of maximum (or minimum) size [17, 20]. In the Roommates case it is NP-complete to determine whether a stable matching exists (even if preference lists are complete) [26].

Here, we are interested in the Marriage and Roommates scenarios, where each participant expresses preferences over some or all of the others, but we focus on matchings that are *popular* rather than stable. A matching M is *popular* if there is no other matching M' with the property that more participants prefer M' to M than prefer M to M'. M is *strongly popular* if, for any other matching M', more participants prefer M to M' than prefer M' to M. These concepts were introduced in the Marriage context by Gärdenfors [12].

Recently, popular matchings have been studied in the context of the so-called House Allocation problem (HA). An instance of HA involves a set of applicants and a set of houses. Each applicant has a strict order of preference over the houses that are acceptable to him, but houses have no preference over applicants. Abraham et al. [1] described an $O(n+m)$ time algorithm to find a popular matching, if one exists, in an instance of HA, where n is the total number of applicants and houses, and m is the total number of acceptable applicant-house pairs. In the case that ties are allowed in the preference lists, they gave an $O(\sqrt{n}m)$ time algorithm. These results motivated the present study.

1.2 The contribution of this paper

Our prime focus in this paper is the problem of finding popular matchings in the Roommates and Marriage contexts. In Section 2 we formalise the problem descriptions and give the necessary terminology and notation. In Section 3 we focus on strict preferences. We describe some basic properties of popular matchings, and the more restrictive strongly popular matchings, and their relation to stable matchings. We give a linear time algorithm to test for and to find a strongly popular matching for Roommates instances without ties. We show that, given a Roommates instance (with or without ties) and a matching M, we can test whether M is popular in $O(\sqrt{n\alpha(n,m)}m\log^{3/2} n)$ time (where α is the inverse Ackermann's function), and in the Marriage case we show how this can be improved to $O(\sqrt{n}m)$ time. This latter result generalises a previous $O(\sqrt{n}m)$ algorithm for the special case where preference lists may include ties and are *symmetric* (i.e., a man m ranks a woman w in kth place if and only if w ranks m in kth place) [28]. In Section 4 we first investigate which of the results of Section 3 can be extended to the case of ties. Then we establish an NP-completeness result for the problem of determining whether a popular matching exists for a Marriage (or Roommates) instance with ties. We conclude with some open problems in Section 5.

1.3 Related work

Gärdenfors [12] introduced the notions of a *(strong) majority 1assignment*, which is equivalent to a (strongly) popular matching in our terminology. He proved that every stable matching is popular in the Marriage case with strict preferences. Also, he showed that a strongly popular matching is stable in the Marriage case, even if there are ties in the preference lists.

The results of Abraham et al. [1] mentioned above led to a number of subsequent papers exploring further aspects and extensions of popular matchings in HA. Manlove and Sng [21] studied the extension in which each house has a *capacity*, the maximum number of applicants that can be assigned to it in any matching, and gave a $O(\sqrt{C}n_1 + m)$ time algorithm for this variant, where C is the sum of the capacities of the houses and n_1 is the number of applicants. Mestre [24] gave a linear time algorithm for a version of the problem in which each applicant has an associated weight. This algorithm, which assumes that all houses have capacity 1, was extended by Sng and Manlove [29] to the case where houses can have non-unitary capacities. Mahdian [19] showed that, for random instances of HA, popular matchings exist with high probability if the number of houses exceeds the number of applicants by a small constant multiplicative factor. Abraham and Kavitha [2] studied a dynamic version of HA allowing for applicants and houses to enter and leave the market, and for applicants to arbitrarily change their preference lists. They showed the existence of a 2-step *voting path* to compute a new popular matching after every such change, assuming that a popular matching exists. McCutchen [22] focused on instances of HA for which no popular matching exists, defining two notions of 'near popularity', and proving that, for each of these, finding a matching that is as near to popular as possible is NP-hard. Huang et al. [15] built upon the work of McCutchen with a study of approximation algorithms in the context of near popularity. Kavitha and Nasre [18] described algorithms to determine an *optimal* popular matching for various interpretations of optimality. McDermid and Irving [23] characterised the structure of the set of popular matchings for an HA instance, and gave efficient algorithms to count and enumerate the popular matchings, and to find several kinds of optimal popular matchings (in the latter case improving on the time complexities of [18]).

In voting theory, a well-established concept of *majority equilibrium* is the following. Let $S = \{1, 2, \ldots, n\}$ be a society of n individuals, and let X be a set of alternatives. Each individual $i \in S$ has a preference order, $\geq_i$, on X. An alternative $x \in X$ is called a *weak Condorcet winner* if for every $y \in X$ distinct of x, $|\{i \in S : x >_i y\}| \geq |\{i \in S : y >_i x\}|$; x is a *strong Condorcet winner* if for every $y \in X$ distinct of x, $|\{i \in S : x >_i y\}| > |\{i \in S : y >_i x\}|$. It is easy to see that if the set of alternatives is the set of all possible matchings of the individuals then a matching is a weak (respectively strong) Condorcet winner if and only if it is popular (respectively strongly popular). Therefore the recent papers of Chen et al. [4, 5, 3], which are concerned with the problems of finding a weak and strong Concordet winner for special graph models, are related to our work.

2 Problem descriptions, terminology and notation

Since the Roommates problem can be seen as an extension of the Marriage problem, we introduce our notation and terminology in the former setting. An instance I of the *Roommates Problem* (RP) comprises a set of *agents* $A = \{a_1, \ldots, a_n\}$. For each agent a_i there is a subset A_i of $A \setminus \{a_i\}$ containing a_i's *acceptable* partners, and a_i has a linear order over A_i, which we refer to as a_i's *preference list.* If a_j precedes a_k in a_i's preference list, we say that a_i *prefers* a_j to a_k. We are also interested in the extension of RP, called the *Roommates Problem with Ties* (RPT), in which preference lists may contain tied entries, so that a_i prefers a_j to a_k if and only if a_j is a strict predecessor of a_k in a_i's preference list. We say that agent a_i is *indifferent* between a_j and a_k if a_j and a_k are tied in his preference list.

An instance I of RP may also be viewed as a graph $G = (A, E)$ where $\{a_i, a_j\}$ forms an edge in E if and only if a_i and a_j are each *acceptable* to the other. We assume that G contains no isolated vertices, and we let $m = |E|$. We refer to G as the *underlying graph* of I. A *matching* in I is a set of disjoint edges in the underlying graph G.

An instance of the *Marriage Problem with Ties* (MPT) may be viewed as an instance of RPT in which the underlying graph G is bipartite. The *Marriage Problem* (MP) is the analogous restriction of RP. In either case, the two sets of the bipartition are known as the *men* and the *women*.

Let I be an instance of RPT. Let $\mathcal{M}$ denote the set of matchings in I, and let $M \in \mathcal{M}$. Given any $a_i \in A$, if $\{a_i, a_j\} \in M$ for some $a_j \in A$, we say that a_i is *matched* in M and $M(a_i)$ denotes a_j, otherwise a_i is *unmatched* in M.

We define the preferences of an agent over matchings as follows. Given two matchings M and M' in $\mathcal{M}$, we say that an agent a_i *prefers* M' *to* M if either (i) a_i is matched in M' and unmatched in M, or (ii) a_i is matched in both M' and M and prefers $M'(a_i)$ to $M(a_i)$. Let $P(M', M)$ denote the set of agents who prefer M' to M, and let $I(M', M)$ be the set of agents who are indifferent between M' and M (i.e., $a_i \in I(M', M)$ if and only if either (i) a_i is matched in both M' and M and either (a) $M'(a_i) = M(a_i)$ or (b) a_i is indifferent between $M'(a_i)$ and $M(a_i)$, or (ii) a_i is unmatched in both M' and M). Then $P(M, M')$, $P(M', M)$ and $I(M', M)$ $(=I(M, M'))$ partition A.

A *blocking pair* with respect to a matching $M \in \mathcal{M}$ is an edge $\{a_i, a_j\} \in E \backslash M$ such that each of a_i and a_j prefers $\{\{a_i, a_j\}\}$ to M. A matching is *stable* if it admits no blocking pair. As observed earlier, an instance of RP or RPT may or may not admit a stable matching, whereas every instance of MP or MPT admits at least one such matching.

Given two matchings M and M' in $\mathcal{M}$, define $D(M, M') = |P(M, M')| - |P(M', M)|$. Clearly $D(M, M') = -D(M', M)$. We say that M is *more popular than* M', denoted $M \succ M'$, if $D(M, M') > 0$. M is *popular* if $D(M, M') \geq 0$ for all matchings $M' \in \mathcal{M}$. Also M is *strongly popular* if $D(M, M') > 0$ for all matchings $M' \in \mathcal{M} \backslash \{M\}$.[1]

Furthermore, for a set of agents $S \subseteq V(G)$, let $P_S(M, M')$ denote the subset of S whose members prefer M to M'. Let $D_S(M, M') = |P_S(M, M')| - |P_S(M', M)|$. We say that $M \succ_S M'$ if $D_S(M, M') > 0$. We will also use the standard notation $M|_S$ for the restriction of a matching M to the set of agents S, where $v \in S$ is considered to be unmatched in $M|_S$ if he is matched in M but $M(v)$ is not in S.

3 The case of strict preferences

In this section we investigate popular matchings in instances of RP and MP where, by definition, every agent's preference list is strictly ordered.

3.1 Relationships between strongly popular, popular and stable matchings

Let $S_1, S_2, \ldots S_k$ be a partition of $V(G)$. Then for any two matchings M and M', $P(M, M') = \cup_{i=1}^{k} P_{S_i}(M, M')$ and $D(M, M') = \sum_{i=1}^{k} D_{S_i}(M, M')$ by definition. We will prove some useful lemmas by using the above identity for two particular partitions.

First, let us consider the component-wise partition of $V(G)$ for the symmetric difference of two matchings M and M'. For each component G_i of $M \oplus M'$ let $C_i = V(G_i)$. We consider the following equation:

$$D(M, M') = \sum_{i=1}^{k} D_{C_i}(M, M') \qquad (1)$$

[1] In fact it is not difficult to see that M is popular if $D(M, M') \geq 0$ for all maximal matchings $M' \in \mathcal{M}$, and M is *strongly popular* if $D(M, M') > 0$ for all maximal matchings $M' \in \mathcal{M} \backslash \{M\}$.

Note that if $|C_i| = 1$ then it must be the case that $D_{C_i}(M, M') = 0$ since the corresponding agent is either unmatched in both M and M' or has the same partner in M and M'.

Lemma 1. *For a given instance of* RP*, a matching M is popular if and only if, for any other matching M', $D_{C_i}(M, M') \geq 0$ for each component G_i of $M \oplus M'$, where $C_i = V(G_i)$.*

Proof. The popularity of M is straightforward from Equation 1, since

$$D(M, M') = \sum_{i=1}^{k} D_{C_i}(M, M') \geq 0.$$

On the other hand, suppose that M is popular but $D_{C_i}(M, M') < 0$ for some $C_i = V(G_i)$ where G_i is a component of $M \oplus M'$. Then $M^* = (M \setminus M|_{C_i}) \cup M'|_{C_i}$ would lead to $D(M^*, M) > 0$, a contradiction. □

A similar statement holds for strongly popular matchings, as we now show.

Lemma 2. *For a given instance of* RP*, a matching M is strongly popular if and only if, for any other matching M', $D_{C_i}(M, M') > 0$ for each component G_i of $M \oplus M'$, where $C_i = V(G_i)$ and $|C_i| \geq 2$.*

Proof. The strong popularity of M is a consequence of Equation 1, i.e.,

$$D(M, M') = \sum_{i=1}^{k} D_{C_i}(M, M') > 0.$$

On the other hand, suppose that M is strongly popular but $D_{C_i}(M, M') \leq 0$ for some C_i where $C_i = V(G_i)$, G_i is a component of $M \oplus M'$ and $|C_i| \geq 2$. Then $M^* = (M \setminus M|_{C_i}) \cup M'|_{C_i}$ would satisfy $D(M^*, M) \geq 0$, a contradiction. □

Now, let M, M' be any two matchings and let $F = M' \setminus M = \{e_1, e_2, \ldots, e_k\}$. Further let $X \subseteq V(G)$ be the set of agents covered by F and let $\bar{X} = V(G) \setminus X$. Considering the partition $\{E_1, E_2, \ldots E_k, \bar{X}\}$, where E_i represents the end vertices of the edge e_i, we have

$$D(M', M) = \sum_{i=1}^{k} D_{E_i}(M', M) + D_{\bar{X}}(M', M). \tag{2}$$

We use the above equation to prove the following lemma.

Lemma 3. *Suppose that we are given an instance of* RP *and two matchings M and M'.*

a) If $M' \succ M$ then M' must contain an edge that is blocking for M.

b) If M is stable then M is popular.

c) If M is stable and M' is popular then M' covers all the vertices that M covers, implying $|M'| \geq |M|$, and $D_{E_i}(M', M) = 0$ for each $e_i \in M' \setminus M$ (i.e., in each pair corresponding to an edge of $M' \setminus M$ exactly one agent prefers M' to M and the other prefers M to M').

Proof. a) We recall that $M' \succ M$ means $D(M', M) > 0$. From the definition of $\bar{X}$ it is obvious that $D_{\bar{X}}(M', M) \leq 0$, since $P_{\bar{X}}(M', M) = \emptyset$. Therefore some other part of the right hand side of Equation 2 must be positive. But $D_{E_i}(M', M)$ is positive if and only if e_i is blocking.

b) This follows immediately from the previous statement.

c) The popularity of M' implies $D(M', M) \geq 0$. As we have seen $D_{\bar{X}}(M', M) \leq 0$, and also $D_{E_i}(M', M) \leq 0$ for each $e_i \in M' \backslash M$ since M is stable. This means that each term of the right hand side of Equation 2 must be equal to 0. But since $P_{\bar{X}}(M', M) = \emptyset$, it follows that $D_{\bar{X}}(M', M) = 0$ if and only if $P_{\bar{X}}(M, M') = \emptyset$, i.e., when M' covers all the vertices that M covers, implying $|M'| \geq |M|$. Furthermore $D_{E_i}(M', M) = 0$ if and only if exactly one agent of E_i prefers M' to M and the other prefers M to M'.

□

We note that the result of Lemma 3(b) was proved by Gärdenfors [12] for MP.

We continue by giving a straightforward connection between strongly popular and popular matchings in RP.

Proposition 4. *Let I be an instance of* RP *and let M be a strongly popular matching in I. Then M is the only popular matching in I.*

Proof. Let $M' \neq M$ be a matching in I. As M is strongly popular, $|P(M, M')| > |P(M', M)|$, so that M' cannot be popular in I. □

Corollary 5. *An instance of* RP *admits at most one strongly popular matching.*

The following proposition was proved by Gärdenfors [12] for MP. Here we generalise the result to the RP context.

Proposition 6. *Let I be an instance of* RP *and let M be a strongly popular matching in I. Then M is stable in I.*

Proof. If M is not stable then let $\{a_i, a_j\}$ be a blocking pair of M. Let M' be a matching formed from M as follows: (i) remove the edge $\{a_i, M(a_i)\}$ if a_i is matched in M, (ii) remove the edge $\{a_j, M(a_j)\}$ if a_j is matched in M, then (iii) add the edge $\{a_i, a_j\}$. Then $|P(M', M)| = 2$ whilst $|P(M, M')| \leq 2$, contradicting the strong popularity of M. Hence M is stable in I. □

Lemma 3(b) and Proposition 6 thus give the following chain of implications involving properties of a matching M in an instance I of RP:

strongly popular ⇒ stable ⇒ popular ⇒ maximal

The following well-known instance of RP introduced by Gale and Shapley [10], illustrates that a popular matching may exist even if the instance does not admit a stable matching. Moreover, a sub-instance of that (as indicated in Example 1) illustrates that a popular matching may not exist.

Example 1. *The preference lists are as follows.*

$$\begin{array}{llll} a_1: & a_2 & a_3 & a_4 \\ a_2: & a_3 & a_1 & a_4 \\ a_3: & a_1 & a_2 & a_4 \\ a_4: & a_1 & a_2 & a_3 \end{array}$$

This instance of RP *has no stable matching, but it admits two popular matchings, namely $M_1 = \{\{a_1, a_4\}, \{a_2, a_3\}\}$ and $M_2 = \{\{a_2, a_4\}, \{a_1, a_3\}\}$. However, we note that after removing a_4 the resulting instance does not admit any popular matching.* □

Every instance of MP admits a stable matching, and hence a popular matching as well. However, Example 2, an MP instance, illustrates that a unique stable matching (which is also a unique popular matching) is not necessarily strongly popular. Therefore Examples 1 and 2 show that the converse to each of the above implications is not true in general.

Example 2. *There are five agents with preference lists as shown.*

$$\begin{array}{llll} m_1: & w_1 & w_3 & w_2 \\ m_2: & w_1 & w_2 & \end{array} \qquad \begin{array}{lll} w_1: & m_1 & m_2 \\ w_2: & m_1 & m_2 \\ w_3: & m_1 & \end{array}$$

There are four maximal matchings in this instance of MP*, namely:*

$$\begin{array}{lll} M_1 & = & \{\{m_1, w_1\}, \{m_2, w_2\}\} \\ M_2 & = & \{\{m_1, w_3\}, \{m_2, w_1\}\} \\ M_3 & = & \{\{m_1, w_3\}, \{m_2, w_2\}\} \\ M_4 & = & \{\{m_1, w_2\}, \{m_2, w_1\}\} \end{array}$$

The following table shows the value of $|P(M_r, M_s)|$ *for each* (r, s) $(1 \leq r, s \leq 4)$*:*

	M_1	M_2	M_3	M_4
M_1	0	3	2	2
M_2	2	0	2	2
M_3	1	1	0	2
M_4	2	1	3	0

M_1 is the unique stable matching. The table entries also indicate that M_1 is the unique popular matching, though M_1 is not strongly popular. □

Furthermore, Examples 3, 4 and 5, presented below, illustrate some interesting properties of MP instances, namely, that a popular matching can be larger than a stable matching, a maximum cardinality matching need not be popular, and the relation $\succ$ can cycle (even if a stable matching exists).

Example 3. *The preference lists for an instance of* MP *are as shown.*

$$\begin{array}{lll} m_1: & w_2 & w_1 \\ m_2: & w_2 & \end{array} \qquad \begin{array}{lll} w_1: & m_1 & \\ w_2: & m_1 & m_2 \end{array}$$

The matching $M_1 = \{\{m_1, w_2\}\}$ is stable (in fact M is the unique stable matching) and thus popular for this instance. However the matching $M_2 = \{\{m_1, w_1\}, \{m_2, w_2\}\}$ is popular but not stable. This example also illustrates that a popular matching can be larger than a stable matching, which motivates the problem of finding a maximum cardinality popular matching, given an instance of MP. □

Example 4. *The preference lists for an instance of* MP *are as shown.*

$$\begin{array}{lll} m_1: & w_1 & \\ m_2: & w_1 & w_2 \\ m_3: & w_2 & w_3 \end{array} \qquad \begin{array}{lll} w_1: & m_2 & m_1 \\ w_2: & m_3 & m_2 \\ w_3: & m_3 & \end{array}$$

The unique perfect matching $M_1 = \{\{m_1, w_1\}, \{m_2, w_2\}, \{m_3, w_3\}\}$ is not popular ($M_2 = \{\{m_2, w_1\}, \{m_3, w_2\})\}$ is more popular than M_1). □

Example 5. *The following instance of* MP *was presented in the seminal paper of Gale and Shapley [10]. The preference lists are as shown.*

$$\begin{array}{llll} m_1: & w_1 & w_3 & w_2 \\ m_2: & w_3 & w_2 & w_1 \\ m_3: & w_2 & w_1 & w_3 \end{array} \qquad \begin{array}{llll} w_1: & m_2 & m_3 & m_1 \\ w_2: & m_1 & m_2 & m_3 \\ w_3: & m_3 & m_1 & m_2 \end{array}$$

Matching $M_0 = \{\{m_1, w_1\}, \{m_2, w_3\}, \{m_3, w_2\}\}$ *is stable and hence popular for this instance. However consider the following matchings:*

$$\begin{aligned} M_1 &= \{\{m_1, w_1\}, \{m_2, w_2\}, \{m_3, w_3\}\} \\ M_2 &= \{\{m_1, w_2\}, \{m_2, w_3\}, \{m_3, w_1\}\} \\ M_3 &= \{\{m_1, w_3\}, \{m_2, w_1\}, \{m_3, w_2\}\} \end{aligned}$$

It is easy to verify that $M_1 \prec M_2 \prec M_3 \prec M_1$. □

3.2 Testing for and finding a strongly popular matching

We begin this section by giving an $O(m)$ algorithm that tests a given stable matching for strong popularity. Let I be an instance of RP and let M be a stable matching in I. Define the graph $H_M = (A, E_M)$, where

$$E_M = \left\{ \{a_i, a_j\} \in E : \begin{array}{l} a_i \text{ is unmatched in } M \text{ or prefers } a_j \text{ to } M(a_i) \vee \\ a_j \text{ is unmatched in } M \text{ or prefers } a_i \text{ to } M(a_j) \end{array} \right\}.$$

Lemma 7. *Let* I *be an instance of* RP *and let* M *be a stable matching in* I*. Let* H_M *be the graph defined above. Then* M *is strongly popular in* I *if and only if* H_M *contains no alternating cycle or augmenting path relative to* M*.*

Proof. Strong popularity implies that no such alternating cycle or augmenting path exists in H_M relative to M by Lemma 2. This is because if M' is the matching obtained by switching edges along this alternating path (or cycle) and C_i denotes the set of agents involved then it would be the case that $D_{C_i}(M, M') \leq 0$ in this component. On the other hand, suppose that M is stable but not strongly popular, i.e., there is a matching M' such that $D(M', M) = 0$. The statements of Lemma 3(c) hold in this case too by the very same argument used in the proof of that result (only with the difference that whilst M was stable and M' was popular in Lemma 3, here M is stable and M' is as popular as M, but not necessarily popular). The fact that M' covers all the vertices that are covered by M means that each component of $M' \oplus M$ is either an alternating cycle or an augmenting path. And since $D_{E_i}(M', M) = 0$ for each $e_i \in M' \setminus M$, every edge in $M' \setminus M$ must belong to H_M. □

Based on Lemma 7 we can give a linear time algorithm for the problem of finding a strongly popular matching as indicated by the following theorem.

Theorem 8. *Given an instance* I *of* RP*, we may find a strongly popular matching or report that none exists in* $O(m)$ *time.*

Proof. We firstly test whether I admits a stable matching in $O(m)$ time [14, Section 4.5.2]. If no such matching exists, I does not admit a strongly popular matching by Proposition 6. Now suppose that I admits a stable matching M. Then I admits a strongly popular matching if and only if M is strongly popular. For, suppose that I admits a strongly popular matching $M' \neq M$. Then M' is certainly popular, and M is popular by Lemma

3(b), a contradiction to to Proposition 4. By Lemma 7, M is strongly popular if and only if H_M contains no augmenting path or alternating cycle relative to M. Clearly H_M has $O(n)$ vertices and $O(m)$ edges. We may test for the existence of each of these structures in $O(m)$ time (see [6, 8] and [7] respectively). ☐

3.3 Testing for popularity

In order to test a matching M in a given instance of RP for popularity, we form a weighted graph H_M as follows. The vertices of H_M are $A \cup A'$, where $A' = \{a'_1, \ldots, a'_n\}$. The edges of H_M are $E \cup E' \cup E''$, where $E' = \{\{a'_i, a'_j\} : \{a_i, a_j\} \in E\}$ and $E'' = \{\{a_i, a'_i\} : 1 \leq i \leq n\}$. For each edge $\{a_i, a_j\} \in E$, we define $\delta_{i,j}$ as follows:

$$\delta_{i,j} = \begin{cases} 0, & \text{if } \{a_i, a_j\} \in M \\ \frac{1}{2}, & \text{if } a_i \text{ is unmatched in } M \text{ or prefers } a_j \text{ to } M(a_i) \\ -\frac{1}{2}, & \text{otherwise} \end{cases}$$

For each edge $\{a_i, a_j\} \in E$, we define the weight of $\{a_i, a_j\}$ in H_M to be $\delta_{i,j} + \delta_{j,i}$. Similarly, for each edge $\{a'_i, a'_j\} \in E'$, we define the weight of $\{a'_i, a'_j\}$ in H_M to be $\delta_{i,j} + \delta_{j,i}$. Finally, for each edge $\{a_i, a'_i\} \in E''$, we define the weight of $\{a_i, a'_i\}$ in H_M to be -1 if a_i is matched in M, and 0 otherwise. It is clear that the weight of each edge belongs to the set $\{-1, 0, 1\}$.

In what follows, given a matching M in G, we define M' to be a matching in H_M such that $M' = \{\{a'_i, a'_j\} : \{a_i, a_j\} \in M\}$.

Lemma 9. *Let I be an instance of* RP *and let M be a matching in I. Let H_M be the weighted graph defined above. Then M is popular if and only if a maximum weight perfect matching in H_M has weight 0.*

Proof. Let M_1 be any matching in I, and let A_{M_1} denote the agents in A who are matched in M_1. Define the matching

$$S(M_1) = M_1 \cup M'_1 \cup \{\{a_i, a'_i\} : a_i \in A \backslash A_{M_1}\}.$$

We claim that $wt(S(M_1)) = D(M_1, M)$, where $wt(M^{\sim})$ is the weight of a matching $M^{\sim}$ in H_M. To show this let $M''_1 = \{\{a_i, a'_i\} : a_i \in A \backslash A_{M_1}\}$. Also let $X = M_1 \backslash M$. Define n_-, n_0, n_+ to be the numbers of edges of weight $-1, 0, 1$ in X respectively. Also define n''_- to be the number of edges of weight -1 in M''_1. Then $wt(S(M_1)) = 2(n_+ - n_-) - n''_-$. Also $|P(M_1, M)| = n_0 + 2n_+$ and $|P(M, M_1)| = n_0 + 2n_- + n''_-$. So $wt(S(M_1)) = D(M_1, M)$ as claimed. Now suppose that a maximum weight perfect matching in H_M has weight 0. Suppose M is not popular. Then there is a matching M_1 such that $D(M_1, M) > 0$. But $wt(S(M_1)) = D(M_1, M)$, a contradiction.

Conversely suppose that M is popular. By the above claim, $wt(S(M)) = D(M, M) = 0$. Suppose that $S(M)$ is not a maximum weight perfect matching in H_M. Let M^* be a perfect matching in H_M such that $wt(M^*) > 0$. Then either $S(M_1)$ or $S(M_2)$ has positive weight, where $M_1 = M^*|_A$ and $M_2 = \{\{a_i, a_j\} : \{a'_i, a'_j\} \in M^*|_{A'}\}$. Hence by the above claim, it follows that either M_1 or M_2 respectively is more popular than M, a contradiction. ☐

Theorem 10. *Given an instance I of* RP *and a matching M in I, we can test whether M is popular in $O(\sqrt{n\alpha(n,m)}m \log^{3/2} n)$ time.*

Proof. Clearly H_M has $O(n)$ vertices and $O(m)$ edges. The current fastest algorithm for finding a maximum weight perfect matching in a weighted graph with weights $\{-1, 0, 1\}$ has complexity $O(\sqrt{n\alpha(n,m)}m \log^{3/2} n)$ [9]. ☐

It is clear that a perfect matching M^* of positive weight exists in H_M if and only if H_M admits an alternating cycle (relative to $S(M)$) of positive weight. It is an open question whether testing for such an alternating cycle is possible in a better running time than finding a maximum weight perfect matching in the general case.

However, this is possible in the MP case. First we observe that if G is bipartite then H_M is also bipartite. Then the problem of finding an alternating cycle of positive weight can be reduced to the problem of finding a directed cycle of positive weight in D_M, where D_M is a directed graph obtained by orienting the edges of H_M in the following way: all the edges of $S(M)$ are directed from the men to the women and all the other edges are directed from the women to the men. The problem of finding a directed cycle of positive weight in a directed graph with weights $\{-1, 0, 1\}$ (or reporting that none exists) can be solved in $O(\sqrt{n}m)$ time by the algorithm of Goldberg [13]. This implies the following result.

Theorem 11. *Given an instance I of* MP *and a matching M in I, we can test whether M is popular in $O(\sqrt{n}m)$ time.*

4 The case of preferences with ties

In this section we consider popular matchings in instances of RPT and MPT.

4.1 Some results extended to the case of ties

It is not hard to see that Proposition 4, Corollary 5 and Proposition 6 continue to hold in the presence of ties. However, one of the key differences is that stability no longer necessarily implies popularity, so that, in particular, it is not necessarily the case that an instance of MPT admits a popular matching. This is illustrated by the following example.

Example 6. *Here we give an instance of* MPT*, involving three men and three women, that admits two popular matchings, but one of its reduced instances does not admit any popular matching. This instance will be used as a gadget in the proof of Theorem 13 below. The preference lists are as follows.*

$$
\begin{array}{llcll}
m_1: & w_1\ w_2\ w_3 & \qquad & w_1: & (m_1\ m_2\ m_3) \\
m_2: & w_1\ w_2\ w_3 & & w_2: & (m_1\ m_2\ m_3) \\
m_3: & w_1\ w_2\ w_3 & & w_3: & m_1\ m_2\ m_3
\end{array}
$$

Here the two popular matchings are $M_{p_1} = \{\{m_1, w_3\}, \{m_2, w_1\}, \{m_3, w_2\}\}$ and $M_{p_2} = \{\{m_1, w_3\}, \{m_2, w_2\}, \{m_3, w_1\}\}$. To show the popularity of M_{p_1}, it is enough to observe that only m_1 and m_3 could get a better partner, but if m_1 gets a better partner then w_3 must be worse off, and if m_3 gets a better partner (i.e., becomes matched to w_1) then m_2 must be worse off. The popularity of M_{p_2} can be proved in a similar way. Finally it is possible to verify that no other matching is popular in this instance.

However, if we remove w_3 then the reduced instance does not admit any popular matching. This is because, relative to any matching M of size 2, two men can always improve (the unmatched man gets w_2, and $M(w_2)$ gets w_1) whilst only one man is worse off ($M(w_1)$ becomes unmatched). □

The algorithm for testing the popularity of a matching in an instance of RP can be extended to the ties case in a natural way, namely by setting $\delta_{i,j}$ to be 0 if $\{a_i, a_j\} \in M$ or if a_i is indifferent between a_j and $M(a_i)$. As a result we will have weights $\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$ in H_M but the technique and the complexity of the popularity checking algorithm (in both the RPT and MPT cases) does not change. On the other hand, the algorithm for finding a strongly popular matching no longer works for the case of ties.

However we can still check the strong popularity of a matching, using a similar technique to that used for popularity checking, in the following way. A matching M is strongly popular if and only if every perfect matching in H_M, excluding $S(M)$, has negative weight. That is, if every perfect matching $M^* \neq S(M)$ in H_M has weight at most $-\frac{1}{2}$. We can reduce this decision problem to another maximum weight perfect matching problem, where $w_\varepsilon(e) = w(e) - \varepsilon = -\varepsilon$ for every edge $e \in S(M)$ (and $w_\varepsilon(e) = w(e)$ for every edge e of H_M not in $S(M)$) where $\varepsilon < \frac{1}{2n}$. Here $S(M)$ is the only perfect matching of weight 0 for w if and only if the maximum weight of a perfect matching in H_M is $-\varepsilon n$ for w_ε.

To facilitate description of some of the subsequent results in this section, we introduce some shorthand notation for variants of the popular matching problem, as follows:

POP-MPT/RPT: the problem of determining whether a popular matching exists, given an instance of MPT/RPT;

PERFECT-POP-MPT/RPT: the problem of determining whether there exists a perfect popular matching, given an instance of MPT/RPT.

4.2 Perfect popular matchings in the Marriage Problem with Ties

We next show that the problem of deciding whether a perfect popular matching exists, given an instance of MPT, is NP-complete.

Theorem 12. PERFECT-POP-MPT *is NP-complete.*

Proof. We reduce from EXACT-MM, that is the problem of deciding, given a graph G and an integer K, whether G admits a maximal matching of size exactly K. EXACT-MM is NP-complete even for subdivision graphs of cubic graphs [25]. Suppose that we are given an instance I of EXACT-MM with the above restriction on a graph $G = (A \cup B, E)$, where $A = \{u_1, \ldots, u_{n_1}\}$ and $B = \{v_1, \ldots, v_{n_2}\}$ satisfying $3n_1 = 3|A| = 2|B| = 2n_2$. We construct an instance I' of PERFECT-POP-MPT with a graph $G' = (U \cup V, E')$, where U and V are referred to as women and men respectively, as follows. Initially we let $U = A$ and $V = B$.

The *proper part* of I' is the exact copy of I such that all neighbours of each agent $u_i \in A$ (and $v_j \in B$) are in a tie in u_i's (and v_j's) preference list. The agents of the proper part are called *proper agents*. For each edge $\{u_i, v_j\} \in E(G)$, we create two vertices, $s_{i,j} \in V$ and $t_{i,j} \in U$ with three edges, $\{u_i, s_{i,j}\}, \{s_{i,j}, t_{i,j}\}, \{t_{i,j}, v_j\}$ in E', where u_i (and v_j) prefers her (his) proper neighbours to $s_{i,j}$ (to $t_{i,j}$) respectively, $s_{i,j}$ prefers u_i to $t_{i,j}$, whilst $t_{i,j}$ is indifferent between $s_{i,j}$ and v_j. Moreover, for a given $u_i \in A$, all agents of the form $s_{i,j}$ such that $\{u_i, v_j\} \in E$ are tied in u_i's list in I', and similarly, for a given $v_j \in B$, all agents of the form $t_{i,j}$ such that $\{u_i, v_j\} \in E$ are tied in v_j's list in I'. We complete the construction by adding two sets of garbage collectors to V and U, namely $X = \{x_1, \ldots, x_{n_1-K}\}$ of size $n_1 - K$ and $Y = \{y_1, \ldots, y_{n_2-K}\}$ of size $n_2 - K$, respectively, such that these sets of agents appear in a tie at the end of each proper agent's list. That is, each $u_i \in A$ has the members of X in a tie at the tail of her list and each $v_j \in B$ has the members of Y in a tie at the tail of his list. The members of the garbage collectors are indifferent between the proper agents.

Suppose first that we have a maximal matching M of size K in I. We shall prove that $M' = M \cup \{\{s_{i,j}, t_{i,j}\} : \{u_i, v_j\} \in E(G)\} \cup \{\{u_{i_k}, x_k\} : u_{i_k} \in A$ is unmatched in $M\} \cup \{\{v_{j_l}, y_l\} : v_{j_l} \in B$ is unmatched in $M\}$ is a perfect popular matching in I'. M' is perfect obviously. We only need to show that M' is popular. Suppose for a contradiction that there exists a matching M^* more popular than M' in I'. Moreover, let M^* be such a matching where $|M^* \oplus M'|$ is minimal.

First we show that it cannot be the case that $\{\{u_i, s_{i,j}\}, \{t_{i,j}, v_j\}\} \subseteq M^*$ for any edge $\{u_i, v_j\} \in E(G)$. We show that if this would be the case then $M^{**} = (M^* \setminus \{\{u_i, s_{i,j}\}, \{t_{i,j}, v_j\}\}) \cup \{\{u_i, v_j\}, \{s_{i,j}, t_{i,j}\}\}$ would also be more popular than M'. However regarding the symmetric difference of M^{**} and M', we get either $M^{**} \oplus M' = (M^* \oplus M') \setminus \{\{u_i, s_{i,j}\}, \{s_{i,j}, t_{i,j}\}, \{t_{i,j}, v_j\}, \{u_i, v_j\}\}$ if $\{u_i, v_j\} \in M'$, or $M^{**} \oplus M' = ((M^* \oplus M') \setminus \{\{u_i, s_{i,j}\}, \{s_{i,j}, t_{i,j}\}, \{t_{i,j}, v_j\}\}) \cup \{u_i, v_j\}$ if $\{u_i, v_j\} \notin M'$, both implying $|M^{**} \oplus M'| < |M^* \oplus M'|$ which is a contradiction to our assumption on the size of $M^* \oplus M'$.

Let $S_{i,j} = \{u_i, v_j, s_{i,j}, t_{i,j}\}$ and $\bar{S}_{i,j} = V(G') \setminus S_{i,j}$. Obviously $D_{\bar{S}_{i,j}}(M^{**}, M') = D_{\bar{S}_{i,j}}(M^*, M')$ so $D_{S_{i,j}}(M^{**}, M') \geq D_{S_{i,j}}(M^*, M')$ would imply $M^{**} \succ M'$.

- if $\{u_i, v_j\} \in M$ then

$$0 = D_{S_{i,j}}(M^{**}, M') > D_{S_{i,j}}(M^*, M') = -1$$

- if both u_i and v_j are matched in M, but not to each other then again we have

$$0 = D_{S_{i,j}}(M^{**}, M') > D_{S_{i,j}}(M^*, M') = -1$$

- if u_i is unmatched and v_j is matched in M then

$$1 = D_{S_{i,j}}(M^{**}, M') = D_{S_{i,j}}(M^*, M') = 1$$

- if u_i is matched and v_j is unmatched in M then again it follows that

$$1 = D_{S_{i,j}}(M^{**}, M') = D_{S_{i,j}}(M^*, M') = 1.$$

Note that both u_i and v_j cannot be unmatched since M is maximal.

To show that M^* cannot be more popular than M', we identify, for each agent that may prefer M^* to M', a corresponding agent who prefers M' to M^*. Since M' is perfect, no garbage collector can prefer M^* to M', and also no $t_{i,j}$ can prefer M^* to M' for any $\{u_i, v_j\} \in E(G)$, obviously. From the above argument it is clear that if an agent $s_{i,j}$ prefers M^* to M' (which can only happen if $\{u_i, s_{i,j}\} \in M^*$) then $t_{i,j}$ must prefer M' to M^* (since $t_{i,j}$ must be unmatched in M^*, as we proved).

Now we show that for any $u_i \in A$ that prefers M^* to M' there exists either a garbage collector $x_{j_k} \in X$ who becomes unmatched in M^* or some $u_{i_k} \in A$ who prefers M' to M^*. Let us consider the alternating path (in $M^* \oplus M'$) starting from u_i who was unmatched in M and therefore matched to some $x_{j_1} \in X$ in M'. If x_{j_1} is unmatched in M^* then we are done, otherwise we continue with $M^*(x_{j_1}) = u_{i_1}$. If u_{i_1} is matched in M and therefore she is matched to a proper agent in M' then we stop since u_{i_1} must prefer M' to M^*, otherwise we continue with $x_{j_2} = M'(u_{i_1})$, and so on. Eventually this path will lead either to some $x_{j_k} \in X$ who was matched in M' but unmatched in M^*, or to some $u_{i_k} \in A$ who was matched in M but matched to a garbage collector in M^* (therefore she prefers M' to M^*). The uniqueness of the correspondence of u_{i_k} to u_i follows by the construction of the alternating path.

Finally we show that for any $v_j \in V$ that prefers M^* to M' there exists either a garbage collector $y_{i_k} \in Y$ who becomes unmatched in M^* or some $v_{j_k} \in B$ who prefers M' to M^*. This is straightforward since again we can consider the alternating path starting from v_j, and continuing with $M'(v_j) = y_{i_1} \in Y$ that will lead either to some $y_{i_k} \in Y$ who was matched in M' but unmatched in M^*, or to some $v_{j_k} \in B$ who was matched in M (therefore he is matched to a proper agent in M') but matched to a garbage collector in

M^*. The uniqueness of the correspondence of v_{j_k} to v_j follows by the construction of the alternating path. This completes the proof of the first direction.

Conversely, suppose that we have a perfect popular matching M' in I'. The popularity implies that $\{s_{i,j}, t_{i,j}\} \subseteq M'$ for each $\{u_i, v_j\} \in E(G)$, since otherwise, if $\{\{u_i, s_{i,j}\}, \{t_{i,j}, v_j\}\} \subseteq M'$ for some $\{u_i, v_j\} \in E(G)$ then $M^* = (M' \setminus \{\{u_i, s_{i,j}\}, \{t_{i,j}, v_j\}\}) \cup \{\{u_i, v_j\}, \{s_{i,j}, t_{i,j}\}\}$ would be more popular than M' (u_i and v_j prefer M^* to M', $s_{i,j}$ prefers M' to M^* and $t_{i,j}$ is indifferent between M^* and M'). Let $M = M'|_{U \cup V}$. M has size K, obviously. To show that M is maximal, suppose for a contradiction that there is an unmatched woman u_i and an unmatched man v_j such that $\{u_i, v_j\} \in E(G)$. In this case $M'(u_i) \in X$ and $M'(v_j) \in Y$ as proved earlier. Therefore $M^* = (M' \setminus \{\{u_i, M'(u_i)\}, \{v_j, M'(v_j)\}, \{s_{i,j}, t_{i,j}\}) \cup \{\{u_i, s_{i,j}\}, \{t_{i,j}, v_j\}\}$ would be more popular than M', since each of u_i, v_j and $s_{i,j}$ each prefers M^* to M', $t_{i,j}$ is indifferent between M^* and M' and only $M'(u_i)$ and $M'(v_j)$ prefer M' to M^*, a contradiction. □

4.3 Popular matchings in the Marriage Problem with ties

Now we show that the problem of deciding whether a popular matching exists, given an instance of MPT, is NP-complete even if we do not restrict our attention to perfect matchings.

Theorem 13. POP-MPT *is NP-complete.*

Proof. Suppose that we are given an instance I of PERFECT-POP-MPT. We show that we can create an instance I' of POP-MPT with an underlying graph G' such that I admits a perfect popular matching if and only if I' admits a popular matching.

Let the *proper part of* I' be exactly the same as I. The *additional part of* I' is an instance of Example 6. Further we add w_3 to the end of each list of the men in the proper part, and conversely, we extend the preference list of w_3 with the men of the proper part by adding them to the tail of her list in an arbitrary order. Let S denote the set of proper agents and let $\bar{S} = V(G') \setminus S$ denote the set of additional agents.

Suppose first that we have a perfect popular matching, M in I. We shall show that $M' = M \cup M_{p_1}$ is popular in I'. Suppose for a contradiction that a matching M^* is more popular than M' in I'. We may also suppose that w_3 is not matched to a proper agent in M^*, because otherwise $M^* \setminus \{\{w_3, M^*(w_3)\}\}$ would also be more popular than M' since w_3 and $M^*(w_3)$ prefer M' to M^* anyway. But in this case, each component of $M' \oplus M^*$ is either in S or in $\bar{S}$ therefore $D(M^*, M') > 0$ and $D_{\bar{S}}(M^*, M') \leq 0$ implies $D_S(M^*, M') > 0$ which means that $M^*|_S$ is more popular than $M = M'|_S$ in I, a contradiction.

Conversely, suppose that M' is popular in I'. First we shall show that either $M_{p_1} \subseteq M'$ or $M_{p_2} \subseteq M'$. If w_3 was unmatched or $M'(w_3)$ was a proper agent then each of the other five additional agents in $\bar{S} \setminus \{w_3\}$ would have been either unmatched or matched to some other member of $\bar{S} \setminus \{w_3\}$. But the restriction of M' to these additional agents cannot be popular for this subset of agents, therefore a matching M^* that is more popular than M' for this subset (i.e. $M^* \succ_{\bar{S} \setminus \{w_3\}} M'$) and which agrees with M' for $S \cup \{w_3\}$ (so $M^*_{S \cup \{w_3\}} = M'|_{S \cup \{w_3\}}$) would be more popular than M'. Further if $M'(w_3)$ is an additional agent then in the additional part either $M'|_{\bar{S}} = M_{p_1}$ or $M'|_{\bar{S}} = M_{p_2}$, since otherwise a matching M^* which satisfies $M^* \succ_{\bar{S}} M'$ and $M^*|_S = M'|_S$ would be more popular than M'.

Now we claim that no proper agent can be unmatched. Suppose for a contradiction that a proper man, say u, is unmatched in M' and without loss of generality, suppose that $M_{p_1} \subseteq M'$. In this case, $M^* = (M' \setminus M_{p_1}) \cup \{\{u, w_3\}, \{m_1, w_2\}, \{m_3, w_1\}\}$ would be more popular than M' since m_1, m_3 and u prefer M^* to M' and only w_3 and m_2 prefer M' to M^*.

The problem of finding a popular matching that is	Marriage instances		Roommates instances	
	strict	with ties	strict	with ties
arbitrary	P [10, 12]	**NPC**	*open*	**NPC**
maximum	*open*	**NPC**	*open*	**NPC**

Table 1: Complexity results for problems of finding popular matchings

Therefore $M = M' \setminus M_{p_1}$ is perfect, and also popular, since $M^{**} \succ M$ in I would imply $M^{**} \cup M_{p_1} \succ M'$ in I'. □

Theorems 12 and 13 imply the NP-completeness of PERFECT-POP-RPT and POP-RPT.

5 Open problems

In this paper we proved that the problem of finding a perfect popular matching (or reporting that none exists) given an MPT instance is NP-hard, and that the problem remains NP-hard even if we merely seek a popular matching (of arbitrary size). However, the complexity of the problem of constructing a maximum cardinality popular matching in an MP instance remains open. The other main open problem is whether finding a popular matching (or reporting that none exists) is possible in polynomial time for an instance of RP. A third open problem is the complexity of finding a strongly popular matching (or reporting that none exists), for an instance of RPT. Finally we remark that the above-mentioned NP-hardness results were established for MPT instances with incomplete lists, and it is open as to whether the same results hold for complete lists.

Our results and the main open problems are summarised in Table 1.

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