Analysing Spatial Properties on Neighbourhood Spaces

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Abstract

We present a bisimulation relation for neighbourhood spaces, a generalisation of topological spaces. We show that this notion, *path preserving bisimulation*, preserves formulas of the spatial logic SLCS. We then use this preservation result to show that SLCS cannot express standard topological properties such as separation and connectedness. Furthermore, we compare the bisimulation relation with standard modal bisimulation and modal bisimulation with converse on graphs and prove it coincides with the latter.

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1 Introduction

The functionality of modern computer systems is increasingly affected by their spatial properties. For example, correctness and efficiency of distributed algorithms depend on the underlying network topology, e.g., whether nodes are reachable, or if there are disconnected components. Furthermore, for cyber-physical systems like autonomous vehicles, spatial aspects are crucial for safe behaviour. To reason about spatial properties, there exist a variety of spatial logics [1] with different kinds of semantics: geometric, directional, topological, or based on structural properties of concurrent processes [8]. However, the analysis of such spatial logics is much less evolved than the analysis of temporal logics like linear temporal logic [20] or computation tree logic [13].

In this paper, we focus on a kind of spatial logics defined on *neighbourhood spaces* also called *Čech closure spaces* [23] or pretopological spaces: a generalisation of topological spaces, where the closure operator is not required to be idempotent. In particular, we analyse the *Spatial Logic on Closure Spaces* (SLCS) introduced by Ciancia et al. [10]. So far, there exists a model-checking algorithm for SLCS, and it has been used for analysis in various application domains such as congestion in bike-sharing applications [12] and bus schedules [9]. An extension of SLCS with distance measuring operators has been used to analyse medical images [3, 6]. However, to the best of our knowledge, no further study of the overall properties of SLCS has been conducted. For example, it is still an open question what its limits of expressivity are. To relate the structural properties of models to a logical language, we follow the standard approach of defining various notions of bisimulations [7] and studying the invariance of SLCS modalities. To that end, we follow ideas of Kurtonina and de Rijke by extending the bisimulations to cover paths [15]. We also employ these bisimulations to study SLCS on two important subclasses of neighbourhood spaces. The first class consists of *topological spaces*, while the latter is the class of *quasi-discrete spaces*, which can be thought of as (possibly infinite) graphs. These classes are non-disjoint, and neither is a subclass of the other. Furthermore, all finite spaces are quasi-discrete.

The investigation of this paper was inspired by recent work of Baryshnikov and Ghrist [5] on a topological approach to the *target counting problem* in sensor networks, the computational task of determining the total number of targets in a region by aggregating the individual counts of each sensor without recording any target identities nor any positional information. Its mathematical formulation depends on having sensor readings over a continuum field of sensors. However, any implementation must occur over a discrete collection of sensors in a given network. This introduces some limitations as several studies have highlighted [19, 16], in particular it is almost impossible to predict the accuracy of the results a given discretisation yields. This shows the need for general notions to rigorously study how properties of interests are preserved across different kind of spaces and provides motivation for this work.

Our contributions in this paper are as follows.

- Definition of bisimulations between neighbourhood models;
- proof that bisimilar points satisfy the same SLCS formulas;
- use of the defined bisimulations to study expressivity of SLCS; and
- comparison of the introduced notions with bisimulations on graphs treated as neighbourhood spaces.

Our article is organised as follows. We begin in Sect. 2 by presenting some preliminary background on neighbourhood spaces. Sect. 3 introduces the main bisimulation relation: path preserving bisimulation. In Sect. 4, we study the properties of this bisimulation on quasi-discrete spaces. Related work is presented in Sect. 5 and we conclude our work in Sect. 6. The full proofs have been moved to the appendix.

2 Neighbourhood Spaces

In this section we recall the notions of neighbourhood spaces and some related results from general topology we will use in this paper. Our main reference is [23]. For additional general results on these topics and for the proofs of the results reported here, we refer the reader to this source.

▶ **Definition 1** (Filter). *Given a set X, a* filter *F on X is a subset of $\mathbb{P}(X)$, such that F is closed under non-empty intersections, whenever $Y \in F$ and $Y \subseteq Z$, then also $Z \in F$, and finally $\emptyset \notin F$. For a set $A \subseteq X$, the filter generated by A is written as $\langle A \rangle$.*

▶ **Definition 2** (Neighbourhood Space). *Let X be a set together with $\eta \subseteq \mathbb{P}(\mathbb{P}(X))$ given by $\eta = \{\eta(x) \mid x \in X\}$, where every $\eta(x)$ is a filter on X and $x \in \bigcap_{N \in \eta(x)} N$. We call η a* neighbourhood system *on X, and $\mathcal{X} = (X, \eta)$ a* neighbourhood space*. For every set $A \subseteq X$, we have the (unique)* interior *and* closure *operators defined as follows.*

$$\mathcal{I}_\eta(A) = \{x \in A \mid A \in \eta(x)\} \qquad \mathcal{C}_\eta(A) = \{x \in X \mid \forall N \in \eta(x) \colon A \cap N \neq \emptyset\}$$

An element $x \in X$ has a minimal neighbourhood *if there exists $N \in \eta(x)$ such that $N \subseteq N'$ for any neighbourhood $N' \in \eta(x)$. We use $N_{min}(x)$ to refer to the minimal neighbourhood* *of x. If each element $x \in X$ has a minimal neighbourhood, then we call $\mathcal{X}$* quasi-discrete*. Finally, if for every element $x \in X$ and any neighbourhood $N \in \eta(x)$, there is a neighbourhood $M \in \eta(x)$, such that for every $y \in M$, we have also that $N \in \eta(y)$, then $\mathcal{X}$ is* topological*.*

▶ **Proposition 3** (Closure Operator ([23] 14 A.1, 14 B.11, 15 A.1, 15 A.2, 26 A.1, 26 A.9))**.** *For any neighbourhood space $\mathcal{X} = (X, \eta)$, the closure operator $\mathcal{C}$ as induced by η satisfies the following properties:*

1. $\mathcal{C}(\emptyset) = \emptyset$
2. $A \subseteq \mathcal{C}(A)$
3. $\mathcal{C}(A \cup B) = \mathcal{C}(A) \cup \mathcal{C}(B)$
4. *If $\mathcal{X}$ is quasi-discrete then, for any set $A \subseteq X$, $\mathcal{C}(A) = \bigcup_{a \in A} \mathcal{C}(\{a\})$.*
5. *If $\mathcal{X}$ is topological, then for any set $A \subseteq X$, $\mathcal{C}(A) = \mathcal{C}(\mathcal{C}(A))$.*

In the work of Čech [23], the properties of Proposition 3 are used to define closure operators, and the equivalences with the corresponding properties of the neighbourhood systems are shown in several theorems. However, since we will use neighbourhoods as the primary entities in the spaces, we choose to demote the closure operators to be derived.

▶ **Definition 4** (Connectedness ([23] 20 B.1))**.** *Let $\mathcal{X} = (X, \eta)$ be a neighbourhood space. Two subsets U and V of X are* semi-separated*, if $\mathcal{C}(U) \cap V = U \cap \mathcal{C}(V) = \emptyset$. A subset U of $\mathcal{X}$ is* connected*, if it is not the union of two non-empty, semi-separated sets. The space $\mathcal{X}$ is connected, if X is connected.*

We also introduce a special kind of neighbourhood space, employed with a linear order.

▶ **Definition 5** (Index Space)**.** *If (I, η) is a connected neighbourhood space and $\leq \subseteq I \times I$ a linear order on I with the bottom element $0 \in I$, then we call $\mathcal{I} = (I, \eta, \leq, 0)$ an* index space*.*

In the following sections, we will often use the concept of continuous function. Generally, we will use the notation $f[A]$ for the image of a set $A \subseteq X$ under a function $f \colon X \to Y$. Similarly, $f^{-1}[B]$ denotes the preimage of a set $B \subseteq Y$.

▶ **Definition 6** (Continuous Function ([23] 16 A.4))**.** *Let $\mathcal{X}_i = (X_i, \eta_i)$ for $i \in \{1, 2\}$ be two neighbourhood spaces. A function $f \colon X_1 \to X_2$ is* continuous*, if for every $x_1 \in X_1$ and every $N_2 \in \eta_2(f(x_1))$, there is a $N_1 \in \eta_1(x_1)$ such that $f[N_1] \subseteq N_2$. Equivalently, since the neighbourhood system of x_1 is upward closed, for every neighbourhood $N_2 \in \eta_2(f(x_1))$, $f^{-1}[N_2] \in \eta_1(x_1)$. We will also write $f \colon \mathcal{X}_1 \to \mathcal{X}_2$.*

Observe that this coincides with the well-known definition of continuous functions on topological spaces. An important connection between connected sets and continuous functions is that the image of a connected set is connected.

▶ **Lemma 7** (Connectedness and Continuity ([23] 20 B.13))**.** *Let $f \colon \mathcal{X}_1 \to \mathcal{X}_2$ be continuous. If a subset X of $\mathcal{X}_1$ is connected, then $f[X]$ is connected.*

Following Ciancia et al. [10], we extend the typical notion of a topological path to neighbourhood spaces.

▶ **Definition 8** (Path)**.** *For an index space $\mathcal{I}$ and a neighbourhood space $\mathcal{X}$, a continuous function $p \colon \mathcal{I} \to \mathcal{X}$ is a* path *on $\mathcal{X}$. If $p(0) = x$, we will also write $p \colon x \rightsquigarrow \infty$ to denote a* path starting in *x.*

This definition includes both quasi-discrete paths and topological paths as given by Ciancia et al. [10]. For example, two typical index spaces are $\mathcal{I} = (\mathbb{R}, \eta_{\mathbb{R}}, \leq, 0)$ with the standard topology based on open intervals, and $\mathcal{I} = (\mathbb{N}, \eta_{\mathbb{N}}, \leq, 0)$, where $\eta_{\mathbb{N}}$ is given by the quasi-discrete neighbourhood system induced by the successor relation. That is, the minimal neighbourhood of each point n is given by $\{n, n+1\}$. Furthermore, observe that by the definition of index spaces and Lemma 7, the image of a path is connected.

We now present spatial models based on neighbourhood spaces and, based on that, the syntax and semantics of SLCS. For the rest of the paper, we let AP be a fixed denumerable set of propositional atoms.

▶ **Definition 9** (Neighbourhood Model). *Let $\mathcal{X} = (X, \eta)$ be a neighbourhood space, $\mathcal{I}$ an index space, and let $\nu \colon X \to \mathbb{P}(AP)$ be a valuation. Then $\mathcal{M} = (\mathcal{X}, \mathcal{I}, \nu)$ is a* neighbourhood model*. We will also write $\mathcal{M} = (X, \eta, \nu)$ to denote neighbourhood models, if the index space is clear from the context.*

We lift all suitable previous definitions to neighbourhood models in the obvious ways. For example, we will speak of continuous functions between the underlying spaces of two models as continuous functions between the models.

▶ **Definition 10** (Syntax of SLCS).

$$\varphi ::= p \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathcal{N}\,\varphi \mid \varphi\,\mathcal{R}\,\varphi \mid \varphi\,\mathcal{P}\,\varphi$$

$\mathcal{N}$ is read as near*, $\mathcal{R}$ is read as* reachable from*, and $\mathcal{P}$ is read as* propagates to*.*

The intuition behind the modalities is as follows. A point satisfies $\mathcal{N}\,\varphi$, if it is contained in the closure of the set of points satisfying φ. Hence, even if it does not satisfy φ itself, it is close to a point that does. A point x is satisfying $\varphi\,\mathcal{R}\,\psi$ if there is a point y satisfying ψ such that x is reachable from y via a path where every point on this path between x and y satisfies φ. Propagation is in a sense the converse modality, i.e., if there is a point y satisfying ψ such that there is a path starting in x and reaching y at some index, and all points in between satisfy φ, then x satisfies $\varphi\,\mathcal{P}\,\psi$. This intuition is formalised in the following semantics.

▶ **Definition 11** (Path Semantics of SLCS). *Let $\mathcal{M} = (\mathcal{X}, \mathcal{I}, \nu)$ be a neighbourhood model and $x \in \mathcal{X}$. The semantics of SLCS with respect to $\mathcal{M}$ is defined inductively as follows.*[1]

$$\begin{array}{lll}
\mathcal{M}, x \models \top & & \text{for all } \mathcal{M} \text{ and } x \\
\mathcal{M}, x \models p & \text{iff} & p \in \nu(x) \\
\mathcal{M}, x \models \neg\varphi & \text{iff} & \text{not } \mathcal{M}, x \models \varphi \\
\mathcal{M}, x \models \varphi \wedge \psi & \text{iff} & \mathcal{M}, x \models \varphi \text{ and } \mathcal{M}, x \models \psi \\
\mathcal{M}, x \models \mathcal{N}\,\varphi & \text{iff} & x \in \mathcal{C}(\{y \mid \mathcal{M}, y \models \varphi\}) \\
\mathcal{M}, x \models \varphi\,\mathcal{R}\,\psi & \text{iff} & \text{there are } y, n \text{ and } p \colon y \rightsquigarrow \infty \text{ such that } p(n) = x \text{ and } \mathcal{M}, y \models \psi \\
& & \quad \text{and for all } 0 < i \leq n \colon \mathcal{M}, p(i) \models \varphi \\
\mathcal{M}, x \models \varphi\,\mathcal{P}\,\psi & \text{iff} & \text{there are } p \colon x \rightsquigarrow \infty \text{ and } n \text{ such that } \mathcal{M}, p(n) \models \psi \\
& & \quad \text{and for all } i \colon 0 \leq i < n \colon \mathcal{M}, p(i) \models \varphi
\end{array}$$

[1] The original definition of the path semantics by Ciancia et al. [10] differs from our presentation. This is due to a change in their definition of the closure operator. In particular, they define the closure on quasi-discrete spaces, i.e., with respect to a given relation R as $\mathcal{C}_R(A) = A \cup \{x \in X \mid \exists a \in A \colon (a, x) \in R\}$. Our definition yields $\mathcal{C}_R(A) = A \cup \{x \in X \mid \exists a \in A \colon (x, a) \in R\}$ (see the discussion at the end of this section), which is more in line with other literature [23, 14]. However, this only changes whether $\mathcal{N}$ can be considered the one-step counterpart of $\mathcal{R}$ or of $\mathcal{P}$.

Ciancia et al. base SLCS on a slightly different set of operators [10]. In particular, they employ a modality $\mathcal{S}$, where $\varphi \, \mathcal{S} \, \psi$ expresses that the current point is within a set satisfying φ that is *surrounded* by a set of points satisfying ψ. However, we chose to have a more symmetric set of operators, and thus use $\mathcal{R}$ instead. This is not problematic, since $\mathcal{S}$ can be expressed by the following equivalence: $(\varphi \, \mathcal{S} \, \psi) \leftrightarrow (\varphi \wedge \neg(\varphi \, \mathcal{R} \, \neg(\varphi \vee \psi)))$.

Let $\mathcal{M} = (\mathcal{X}, \mathcal{I}, \nu)$ be a model, and p a path $p \colon x \rightsquigarrow \infty$ in $\mathcal{M}$. For $n, m \in \mathcal{I}$ and $n < m$, we use (n, m) as notation for the set $\{i \mid n < i < m\}$, similar to the usual notation of open intervals on the indexspace $\mathcal{I}$. For such an interval (n, m) and an SLCS formula φ, we use the following abbreviation to denote the *satisfaction of* φ *within* (n, m):

$$\mathcal{M}, p, (n, m) \models \varphi \text{ iff for all } i \text{ with } n < i < m \text{ we have } \mathcal{M}, p(i) \models \varphi \; .$$

With this notation, the semantics of $\mathcal{R}$ and $\mathcal{P}$ read as follows.

$$\begin{aligned} \mathcal{M}, x \models \varphi \, \mathcal{R} \, \psi \text{ iff } & \exists p \colon y \rightsquigarrow \infty \text{ and } n \text{ s.t. } p(n) = x, \mathcal{M}, y \models \psi, \mathcal{M}, x \models \varphi, \\ & \text{and } \mathcal{M}, p, (0, n) \models \varphi \\ \mathcal{M}, x \models \varphi \, \mathcal{P} \, \psi \text{ iff } & \exists p \colon x \rightsquigarrow \infty \text{ and } n \text{ s.t. } \mathcal{M}, p(n) \models \psi, \mathcal{M}, x \models \varphi, \text{ and } \mathcal{M}, p, (0, n) \models \varphi \end{aligned}$$

While we are able to define SLCS for the setting of general neighbourhood models, we will often restrict our attention to one of the following two special cases: quasi-discrete and topological models. They are defined as follows.

▶ **Definition 12** (Quasi-Discrete and Topological Models)**.** *Let $\mathcal{X}$ be a quasi-discrete neighbourhood space, and $\mathcal{I}_{\mathbb{N}} = (\mathbb{N}, \eta_{\mathbb{N}}, \leq, 0)$ be the index space defined by the natural numbers. Then a model $\mathcal{M} = (\mathcal{X}, \mathcal{I}_{\mathbb{N}}, \nu)$ based on these spaces is a* quasi-discrete neighbourhood model. *Similarly, if $\mathcal{X}$ is topological, and $\mathcal{I}_{\mathbb{R}} = (\mathbb{R}, \eta_{\mathbb{R}}, \leq, 0)$ is the index space defined by the real numbers, and the topology based on all open intervals as well as the standard ordering of the reals, a model $\mathcal{M} = (\mathcal{X}, \mathcal{I}_{\mathbb{R}}, \nu)$ is a* topological neighbourhood model.

Hence, whenever we refer to a model as quasi-discrete, we fix the index space to the natural numbers, and similarly, whenever a model is topological, we only allow for topological paths. Observe that every quasi-discrete space can be described as a (possibly infinite) graph structure. For a quasi-discrete space (X, η) the induced edge relation $R \subseteq X \times X$ is defined as $\{(x, y) \mid y \in N_{min}(x)\}$. This results in the closure operator being defined on points of a quasi-discrete space as $\mathcal{C}(x) = \{y \in X \mid x \in N_{min}(y)\}$. Furthermore, as $x \in N_{min}(x)$ for any $x \in X$, it follows that R is reflexive (as also shown in [23] 26 A.2). On the other hand, every graph $G = (V, R)$ (where $R \subseteq V \times V$ is not necessarily reflexive) induces a quasi-discrete space, by setting the minimal neighbourhood of a vertex $x \in V$ to be $N_{min}(x) = \{x\} \cup \{y \mid (x, y) \in R\}$. Whenever we depict quasi-discrete models as graphs, we will omit the implicit loops on nodes.

Of course, there are neighbourhood spaces that are both quasi-discrete *and* topological. This is the case if the edge relation of the graph representation of a quasi-discrete space is transitive (see [23], Theorem 26 A.2). In particular, fully connected bidirectional graphs are also topological, if considered as neighbourhood spaces. For such spaces, we have to restrict ourselves to treat them either as topological or as quasi-discrete.

3 Bisimulations for Neighbourhood Spaces

In this section we define two notions of bisimulation for neighbourhood spaces: *neighbourhood bisimulation* and *path preserving bisimulation*. We will then use them to study the preservation of SLCS formulas across models and thus the expressivity of SLCS.

▶ **Definition 13** (Neighbourhood Bisimulation). *Let (X_1, η_1, ν_1) and (X_2, η_2, ν_2) be two neighbourhood models over the same index space, and $x_1 \in X_1$, $x_2 \in X_2$ two points of the respective models. A relation $Z_\eta \subseteq X_1 \times X_2$ with $x_1 Z_\eta x_2$ is a* neighbourhood bisimulation *of x_1 and x_2, if we have*

(atm) *$p \in \nu_1(x_1)$ if, and only if, $p \in \nu_2(x_2)$ for all $p \in AP$*

(frt$_\eta$) *for every neighbourhood $N_2 \in \eta_2(x_2)$, there is a neighbourhood $N_1 \in \eta_1(x_1)$ such that for all $y_1 \in N_1$, there is a $y_2 \in N_2$ with $y_1 Z_\eta y_2$*

(bck$_\eta$) *for every neighbourhood $N_1 \in \eta_1(x_1)$, there is a neighbourhood $N_2 \in \eta_2(x_2)$ such that for all $y_2 \in N_2$, there is a $y_1 \in N_1$ with $y_1 Z_\eta y_2$*

Two models $\mathcal{M}_1$ and $\mathcal{M}_2$ are neighbourhood bisimilar *at x_1 and x_2, if there is a neighbourhood bisimulation Z_η such that $x_1 Z_\eta x_2$.*

We can prove that SLCS formulas using only the "near" modality are invariant under neighbourhood bisimulation. While we do not present a separate theorem for this fact due to space reasons, its proof can be extracted from the corresponding induction step of the proof of Theorem 17.

▶ **Example 14.** Let $\mathcal{M}_\mathbb{R} = ((\mathbb{R}, \eta_\mathbb{R}), \mathcal{I}_\mathbb{R}, \nu_\mathbb{R})$ be a topological neighbourhood model, where the underlying space is given by the usual topology on the real numbers, and $\nu_\mathbb{R}(s) = \{a\}$ for all $s \in (-1, 1)$ and $\nu_\mathbb{R}(s) = \emptyset$ otherwise. Furthermore, let $\mathcal{M}_2 = ((\{x, y\}, \eta_2), \mathcal{I}_\mathbb{R}, \nu_2)$ be a topological model where η_2 is the discrete topology on the set $\{x, y\}$ (i.e., $N_{min}(x) = \{x\}$ and $N_{min}(y) = \{y\}$), $\nu_2(x) = \{a\}$, and $\nu_2(y) = \emptyset$. Then the relation Z_η, given by $sZ_\eta x$ for all $s \in (-1, 1)$, is a neighbourhood bisimulation between any point $s \in (-1, 1)$ and x.

Observe that it is not total, and in particular, there cannot be a total neighbourhood bisimulation between these two spaces: If there was, it would need to relate 1 to y, since neither satisfies any proposition, and y is the only such point in $\mathcal{M}_2$. However, consider the neighbourhood $\{y\} \in \eta_2(y)$. Every neighbourhood of 1 contains a point $s < 1$, which is not in relation with y. Hence, there is no neighbourhood N of 1 such that every element of N is in relation with an element of $\{y\}$.

In the preceeding example, all points that are related by Z_η indeed satsify the same formulas using only $\mathcal{N}$, in this case Boolean combinations of the formulas $\mathcal{N}\, a$ and $\neg \mathcal{N} \neg a$ (or equivalent formulas). However, $\mathcal{M}_\mathbb{R}, 0 \models a\, \mathcal{P} \neg a$, while $\mathcal{M}_2, x \not\models a\, \mathcal{P} \neg a$. To ensure the preservation of formulas using the path modalities $\mathcal{P}$ and $\mathcal{R}$, we strengthen our notion of bisimulation following ideas of Kurtonina and de Rijke [15]. Specifically, we not only need points in the two models to be related, but also intervals over paths. This is achieved by introducing two relations Z_1 and Z_2, the former relating path intervals from the first model to the second, and the latter the other way around. The resulting bisimulation is based on a triple or relations (Z_η, Z_1, Z_2) and defined as follows.

▶ **Definition 15** (Path Preserving Bisimulation). *Let $\mathcal{M}_1 = ((X_1, \eta_1), \mathcal{I}, \nu_1)$ and $\mathcal{M}_2 = ((X_2, \eta_2), \mathcal{I}, \nu_2)$ be two neighbourhood models over the same index space $\mathcal{I}$, and $\mathcal{P}$ and $\mathcal{Q}$ sets of all possible paths on $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. A* path preserving bisimulation *between $\mathcal{M}_1$ and $\mathcal{M}_2$ is triple (Z_η, Z_1, Z_2), where $Z_\eta \subseteq X_1 \times X_2$, Z_1 a relation between $\mathcal{P} \times \mathcal{I}$ and $\mathcal{Q} \times \mathcal{I}$, and Z_2 a relation between $\mathcal{Q} \times \mathcal{I}$ and $\mathcal{P} \times \mathcal{I}$ s.t. $Z_\eta \neq \emptyset$ and the following holds for all $x_1 \in X_1$, $x_2 \in X_2$, $(p, n) \in \mathcal{P} \times \mathcal{I}$ and $(q, m) \in \mathcal{Q} \times \mathcal{I}$.*

1. *if $x_1\ Z_\eta\ x_2$, then Z_η is a neighbourhood bisimulation;*
2. *if $x_1\ Z_\eta\ x_2$, $p\colon x_1 \rightsquigarrow \infty$ and $n \neq 0$, then there exists $q\colon x_2 \rightsquigarrow \infty$ and m s.t. $p(n)\ Z_\eta\ q(m)$ and $(p, n)\ Z_1\ (q, m)$;*
3. *if $x_1\ Z_\eta\ x_2$, $p\colon y_1 \rightsquigarrow \infty$ with $p(n) = x_1$ and $n \neq 0$, then there exists $q\colon y_2 \rightsquigarrow \infty$ and m with $q(m) = x_2$ s.t. $p(0)\ Z_\eta\ q(0)$ and $(p, n)\ Z_1\ (q, m)$;*

4. *if* $(p, n)\ Z_1\ (q, m)$ *and there exists* $k_q \in \mathcal{I}$ *with* $0 < k_q < m$*, then there exists* $k_p \in \mathcal{I}$ *with* $0 < k_p < n$ *s.t.* $p(k_p)\ Z_\eta\ q(k_q)$*;*
5. *if* $x_1\ Z_\eta\ x_2$*,* $q\colon x_2 \rightsquigarrow \infty$ *and* $m \neq 0$*, then there exists* $p\colon x_1 \rightsquigarrow \infty$ *and* n *s.t.* $p(n)\ Z_\eta\ q(m)$ *and* $(q, m)\ Z_2\ (p, n)$*;*
6. *if* $x_1\ Z_\eta\ x_2$*,* $q\colon y_2 \rightsquigarrow \infty$ *with* $q(m) = x_2$ *and* $m \neq 0$*, then there exists* $p\colon y_1 \rightsquigarrow \infty$ *and* n *with* $p(n) = x_1$ *s.t.* $p(0)\ Z_\eta\ q(0)$ *and* $(q, m)\ Z_2\ (p, n)$*; and*
7. *if* $(q, m)\ Z_2\ (p, n)$ *and there exists* $k_p \in \mathcal{I}$ *with* $0 < k_p < n$*, then there exists* $k_q \in \mathcal{I}$ *with* $0 < k_q < m$ *s.t.* $p(k_p)\ Z_\eta\ q(k_q)$*.*

It is straightforward to show that for three models $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$ over the same index space $\mathcal{I}$, whenever there is a path preserving bisimulation between $x_1 \in \mathcal{M}_1$ and $x_2 \in \mathcal{M}_2$, and there is a path preserving bisimulation between x_2 and $x_3 \in \mathcal{M}_3$, then there is also a path preserving bisimulation between x_1 and x_3.

Before we show that the truth of all SLCS formulas is preserved under path preserving bisimulation, we first present the following technical lemma.

▶ **Lemma 16.** *Let* (Z_η, Z_1, Z_2) *be a path preserving bisimulation between* $\mathcal{M}_1$ *and* $\mathcal{M}_2$*, and* φ *be an SLCS formula that is invariant under neighbourhood bisimulation, i.e., for any* $x_1 \in \mathcal{M}_1$ *and* $x_2 \in \mathcal{M}_2$ *with* $x_1 Z_\eta x_2$*, we have* $\mathcal{M}_1, x_1 \models \varphi$ *if, and only if,* $\mathcal{M}_2, x_2 \models \varphi$*. For two paths* p *and* q *with* $(p, n)\ Z_1\ (q, m)$*, we have* $\mathcal{M}_1, p, (0, n) \models \varphi$ *implies* $\mathcal{M}_2, q, (0, m) \models \varphi$*. Additionally, if* $(q, m) Z_2 (p, n)$ *then* $\mathcal{M}_2, q, (0, m) \models \varphi$ *implies* $\mathcal{M}_1, p, (0, n) \models \varphi$*.*

▶ **Theorem 17.** *If* (Z_η, Z_1, Z_2) *is a path preserving bisimulation between* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *with* $x_1 Z_\eta x_2$*, then* $\mathcal{M}_1, x_1 \models \varphi$ *if, and only if,* $\mathcal{M}_2, x_2 \models \varphi$ *for every formula* φ *of SLCS.*

Proof. We proceed by induction on the length of formulas. The induction base and the cases for the Boolean operators are as usual. For the *near* modality, the induction step consists basically of a straightforward application of the definitions. We provide a sketch for the preservation of *propagate*. The case for *reachable* is analogous.

So let $\mathcal{M}_1, x_1 \models \varphi \mathcal{P} \psi$. That is, there is a path p starting in x_1 and visiting a point satisfying ψ at the index n, where all points in between satisfy φ. By the bisimulation property (Def. 15 (2)), there is a path q starting in x_2 that visits, at m, a point that is bisimilar to $p(n)$, and for all indices between 0 and m, there are bisimilar points on p as well. Hence, by the induction hypothesis and Lemma 16, q is a witness that $\mathcal{M}_2, x_2 \models \varphi \mathcal{P} \psi$. The other direction is similar, using the second case of Lemma 16. ◀

Note that we do not show that logical equivalence of two points implies that they are bisimilar (cf. Sect. 6). Now that we have a suitable notion of bisimilarity, we can use it to analyse whether SLCS is able to capture spatial properties. As an example, we show that SLCS is neither capable of expressing standard topological separation axioms nor the connectedness of a model.

▶ **Definition 18** (Separation Properties)**.** *Let* $\mathcal{X}$ *be a neighbourhood space. If for every two points* $x, y \in \mathcal{X}$ *we have that* $y \in \mathcal{C}(\{x\})$ *and* $x \in \mathcal{C}(\{y\})$ *implies* $x = y$*, then* $\mathcal{X}$ *is* T_0-separated. *If* $\{x\} \cap \mathcal{C}(y) = \mathcal{C}(x) \cap \{y\} = \emptyset$ *for all distinct* x *and* y*, then* $\mathcal{X}$ *is* T_1*-separated.*[2] *We call a neighbourhood model* T_i*-separated, if its underlying space is* T_i*-separated for* $i \in \{0, 1\}$*.*

▶ **Proposition 19.** *There is no formula of SLCS expressing* T_0 *separation.*

[2] Čech calls such spaces *feebly semi-separated* and *semi-separated*, respectively, [23], but the name T_0 and T_1 for these properties are standard in topology.

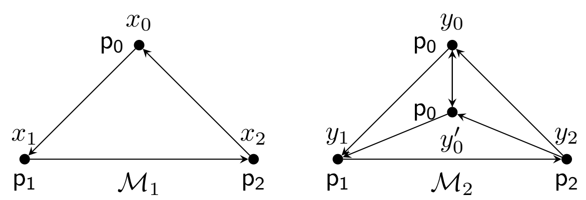

Figure 1 $\mathcal{M}_1$ is T_0-separated, but $\mathcal{M}_2$ is not.

Proof. Consider the quasi-discrete models $\mathcal{M}_1$ and $\mathcal{M}_2$ in Fig. 1, and the relation Z_η given by $x_i Z_\eta y_i$ and $x_0 Z_\eta y_0'$, where Z_1 is defined by $(p, n)Z_1(q, n)$ iff $p(0)Z_\eta q(0)$ and

$$p(i) = x_0 \Leftrightarrow q(i) \in \{y_0, y_0'\} \ ,$$
$$p(i) = x_1 \Leftrightarrow q(i) = y_1 \ ,$$
$$p(i) = x_2 \Leftrightarrow q(i) = y_2 \ .$$

The relation Z_2 is then given by $Z_2 = Z_1^{-1}$. Then the triple of these three relations is a path preserving bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$. For example, consider the minimal neighbourhood $N_{min}(x_1) = \{x_1, x_2\}$ of x_1. Then choose $N_{min}(y_1) = \{y_1, y_2\}$ as a neighbourhood of y_1. For every element of $N_{min}(y_1)$, there is an element in $N_{min}(x_1)$, such that the elements are bisimilar. The other neighbourhoods can be checked similarly. So, all points in $\mathcal{M}_1$ and $\mathcal{M}_2$ satisfy the same set formulas of SLCS by Theorem 17. But it is also easy to check that $\mathcal{M}_1$ is T_0-separated, while $\mathcal{M}_2$ is not. Hence no formula of SLCS expresses T_0-separation. ◀

▶ **Proposition 20.** *There is no formula of SLCS expressing T_1 separation.*

Proof. Let X be an uncountable set. Let $\mathcal{Y}$ be the set of all subsets of X, such that for every $Y \in \mathcal{Y}$, either $Y = \emptyset$, or the complement of Y is countable. Then, for every $x \in X$, let $\eta_1(x) = \{N \mid \exists Y \in \mathcal{Y} \colon Y \subseteq N \wedge x \in Y\}$. Then $\mathcal{X} = (X, \eta_1)$ is called the *countable complement topology*. For any valuation ν_1 over X, $\mathcal{M}_1 = (\mathcal{X}_1, \mathcal{I}_\mathbb{R}, \nu_1)$ is a topological model. Also, let X' be constructed from X by "doubling" all points, i.e., $X' = \{x' \mid x \in X\} \cup X$, where each x' is a new, distinct, element to the x it is constructed from. Then, let $\mathcal{Y}'$ be the doubling of every set in $\mathcal{Y}$ in a similar way, and η_2 be defined similar to η_1, but over $\mathcal{Y}'$. Then, $\mathcal{X}_2 = (X', \eta_2)$ is the *double pointed countable complement topology*. Also, let ν_2 be the valuation that assigns the value of $\nu_1(x)$ to each x and x'. Then, $\mathcal{M}_2 = (\mathcal{X}_2, \mathcal{I}_\mathbb{R}, \nu_2)$ is also a topological model.

The relation given by $xZ_\eta y$ iff $y = x \vee y = x'$ is obviously a neighbourhood bisimulation. Furthermore, we define $(p, n)Z_1(q, m)$ iff $p(0)Z_\eta q(0)$ and $p(i) = z$ iff $q(i) \in \{z, z'\}$, as well as $Z_2 = Z_1^{-1}$. This triple then represents a path preserving bisimulation between the two models. However, $\mathcal{M}_1$ is both T_0 and T_1 separated, while $\mathcal{M}_2$ is neither [21]. ◀

▶ **Proposition 21.** *There is no formula of SLCS that is expressing connectedness.*

Proof. Consider an arbitrary neighbourhood model $\mathcal{M}$ and a model $\mathcal{M}'$ consisting of two unconnected copies of $\mathcal{M}$. Then we can define a path preserving neighbourhood bisimulation by relating every point of $\mathcal{M}$ with both of its copies in $\mathcal{M}'$, and every path of $\mathcal{M}$ with both corresponding paths in $\mathcal{M}'$. ◀

Similarly, we can ask whether quasi-discrete models, where the underlying space is also topological, are only bisimilar to other models, where the space is topological. As the

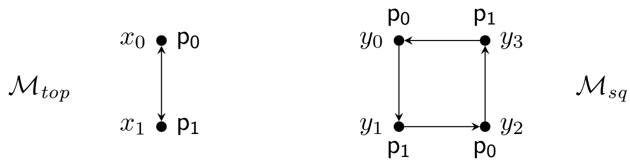

Figure 2 Two bisimilar quasi-discrete models, where $\mathcal{M}_{top}$ is topological and $\mathcal{M}_{sq}$ is not.

next lemma shows, the answer to this question is negative. Hence, SLCS cannot express transitivity of the underlying edge relation.

▶ **Lemma 22.** *There are quasi-discrete models $\mathcal{M}_1$ and $\mathcal{M}_2$ that are bisimilar to each other, and where the underlying space of $\mathcal{M}_1$ is topological, while the space of $\mathcal{M}_2$ is not.*

Proof. Consider the graphs in Fig. 2. If we set $x_i Z_\eta y_j$ iff $j \mod 2 = i$, and relate paths in the obvious way, then we have a path preserving bisimulation. However, $\mathcal{M}_{top}$ is topological, while $\mathcal{M}_{sq}$ is not. ◀

The next example shows that a topological model can be in bisimulation with non-topological models in a non-trivial way. To that end, we exploit the transitivity of models being path preserving bisimilar, by first showing that a specific topological model is path preserving bisimilar to a topological model with an underlying quasi-discrete space, and then show that this second model is path preserving bisimilar to a model over topological paths, but where the underlying space is quasi-discrete, but not topological.

▶ **Example 23.** Let $\mathcal{M} = (\mathcal{X}_2, \mathcal{I}_\mathbb{R}, \nu_2)$ be the topological model based on the double pointed countable complement topology (cf. the proof of Proposition 20), where $\nu_2(x) = \{p_0\}$ and $\nu_2(x') = \{p_1\}$ for any point x of the underlying set. Furthermore, consider the models depicted in Fig. 2, but considered over the index space $\mathcal{I}_\mathbb{R}$. We will first proceed to define a path preserving bisimulation between $\mathcal{M}$ and $\mathcal{M}_{top}$.

Let $x Z_\eta x_0$ and $x' Z_\eta x_1$ for all x of the underlying set of $\mathcal{M}$. Then clearly Z_η is a neighbourhood bisimulation, since any neighbourhood in $\mathcal{M}$ contains both points x and x' and similarly, any neighbourhood in $\mathcal{M}_{top}$ contains both x_0 and x_1.

Now let p be any path on $\mathcal{M}$. Then q defined by $q(i) = x_0$ if $p(i) \in X$ and $q(i) = x_1$ if $p(i) \in X'$, is a path as well, since any function into $\mathcal{M}_{top}$ is continuous (as it possesses the indiscrete topology, that is, for both x_0 and x_1, $\{x_0, x_1\}$ is their only neighbourhood). So, we set $(p, m) Z_1 (q, m)$ for any path, $m \in \mathbb{R}$ and q defined as above. Hence, whenever there is a $0 < k_q < m$, then $p(k_q) Z_\eta q(k_q)$.

Finally, consider a path q on $\mathcal{M}_{top}$. Choose an arbitrary point $x \in X$, and define p by $p(i) = x$ if $q(i) = x_0$ and $p(i) = x'$ if $q(i) = x_1$. Then set $(q, m) Z_2 (p, m)$ for every $m \in \mathbb{R}$. Again, the bisimulation condition is satisfied.

All in all, we have defined a path preserving bisimulation between $\mathcal{M}$ and $\mathcal{M}_{top}$, where every point of $\mathcal{M}$ is bisimilar to either x_0 or x_1.

Now we define a path preserving bisimulation between $\mathcal{M}_{top}$ and $\mathcal{M}_{sq}$. As can be easily checked, the relation $Z_\eta = \{(x_0, y_0), (x_0, y_2), (x_1, y_1), (x_1, y_3)\}$ constitutes a neighbourhood bisimulation. The relation Z_2 can be defined as follows: for any path q on $\mathcal{M}_{sq}$ and $i \in \mathbb{R}$, set $p(i) = x_0$ if $q(i) \in \{y_0, y_2\}$ and $p(i) = x_1$ otherwise. Then p is continuous, since any function into $\mathcal{M}_{top}$ is continuous, and also for any index i, we have $p(i) Z_\eta q(i)$. Hence, we set $(q, m) Z_2 (p, m)$ for any $m \in \mathbb{R}$. For Z_1, let p be a path starting in x_0 and $m \in \mathbb{R}$. Then we define q as

$$q(i) = \begin{cases} y_0 & \text{, if } i < 1 \\ y_3 & \text{, if } 1 \leq i < 2 \\ y_2 & \text{, if } 2 \leq i < 3 \\ y_1 & \text{, if } 3 \leq i \end{cases}$$

Now, we distinguish several cases:

1. if $p(m) = x_0$ and for all $i < m$, $p(m) = x_0$, then $(p,m)Z_1(q,0.5)$,
2. if $p(m) = x_1$ and for all $i < m$, $p(m) = x_0$, then $(p,m)Z_1(q,1)$,
3. if $p(m) = x_0$ and for all $i < m$, $p(m) = x_1$, then $(p,m)Z_1(q,2)$,
4. if $p(m) = x_1$ and for all $i < m$, $p(m) = x_1$, then $(p,m)Z_1(q,1.5)$,
5. if $p(m) = x_0$, for some $i < m$, $p(m) = x_0$ and for some $i < m$, $p(m) = x_1$, then $(p,m)Z_1(q,2.5)$, and
6. if $p(m) = x_1$, for some $i < m$, $p(m) = x_0$ and for some $i < m$, $p(m) = x_1$, then $(p,m)Z_1(q,3.5)$.

For any path with $p(0) = x_1$, we can define a path q in a similar way. It is easy to check that this relation also satisfies the conditions for a path preserving bisimulation.

4 Bisimulations on Quasi-Discrete Spaces

In this section we show how the notions of bisimulation presented in Sect. 3 relate to common notions of bisimulation for modal logic when the models taken into considerations are quasi-discrete neighbourhood models. While being inspired by the bisimulation of Kurtonina and and de Rijke [15], we obtain a different result when comparing the path preservering bisimulation and a bisimulation for modal logic with converse modalities.

Our notions of bisimulation for quasi-discrete neighbourhood models are based on the induced edge relation R_i as described in Sect. 2, and we will refrain in mentioning the underlying index space to ease the notation. As our first notion of bisimulation coincides with the standard notion of bisimulation for modal logic (e.g.,[7]), we refer to it as modal bisimulation.

▶ **Definition 24** (Modal Bisimulation)**.** *Let $\mathcal{M}_1 = (X_1, \eta_1, \nu_1)$ and $\mathcal{M}_2 = (X_2, \eta_2, \nu_2)$ be two quasi-discrete neighbourhood models. A relation $\rho \subseteq X_1 \times X_2$ is a* modal bisimulation*, if for every pair $x_1 \, \rho \, x_2$ the following three conditions hold.*
(atm) *$p \in \nu_1(x_1)$ if, and only if, $p \in \nu_2(x_2)$ for all $p \in AP$*
(frt$_f$) *if $(x_1, y_1) \in R_1$, then there exists $y_2 \in X_2$ with $(x_2, y_2) \in R_2$ and $y_1 \, \rho \, y_2$*
(bck$_f$) *if $(x_2, y_2) \in R_2$, then there exists a $y_1 \in X_1$ with $(x_1, y_1) \in R_1$ and $y_1 \, \rho \, y_2$*

Lemma 25 shows the relationship between modal bisimulation and neighbourhood bisimulation on quasi-discrete neighbourhood models.

▶ **Lemma 25.** *On quasi-discrete neighbourhood models, neighbourhood bisimulation and modal bisimulation coincide.*

In contrast with its behaviour on general neighbourhood spaces, neighbourhood bisimulation on quasi-discrete neighbourhood models preserves the "propagate to" operator.

▶ **Theorem 26.** *If ρ is a modal bisimulation between two quasi-discrete neighbourhood models $\mathcal{M}_1$ and $\mathcal{M}_2$ with $x_1 \; \rho \; x_2$, then $\mathcal{M}_1, x_1 \models \varphi$ if, and only if, $\mathcal{M}_2, x_2 \models \varphi$ for every formula φ of SLCS without $\mathcal{R}$.*

To see that modal bisimulation does not preserve "reachable from", it is enough to consider a very simple example where $\mathcal{M}_1$ is a model composed of a single point x with valuation $\nu_1(x) = \{p\}$, and $\mathcal{M}_2$ is composed of two points $\{y_1, y_2\}$ where $N_{min}(y_1) = \{y_1, y_2\}$, $N_{min}(y_2) = \{y_2\}$, $\nu_2(y_1) = \{q\}$ and $\nu_2(y_2) = \{p\}$. It is easy to note that x and y_2 are modal bisimilar, but "reachable from" is not preserved. The preservation of such an operator would require a backward preservation of paths. This, from a modal logic perspective, corresponds to a notion of bisimulation able to preserve a modal language with converse modalities.

▶ **Definition 27** (Modal Bisimulation with Converse). *Let $\mathcal{M}_1 = (X_1, \eta_1, \nu_1)$ and $\mathcal{M}_2 = (X_2, \eta_2, \nu_2)$ be two quasi-discrete neighbourhood models. A relation $\rho \subseteq X_1 \times X_2$ is a* modal bisimulation with converse*, if it is a modal bisimulation and for every pair $x_1 \,\rho\, x_2$ the following additional conditions hold.*

(frt$_c$) *if $(y_1, x_1) \in R_1$, then there exists $y_2 \in X_2$ with $(y_2, x_2) \in R_2$ and $y_1 \,\rho\, y_2$*

(bck$_c$) *if $(y_2, x_2) \in R_2$, then there exists a $y_1 \in X_1$ with $(y_1, x_1) \in R_1$ and $y_1 \,\rho\, y_2$*

▶ **Lemma 28.** *On quasi-discrete neighbourhood models, path preserving bisimulation and modal bisimulation with converse coincide.*

Lemma 28 differs from results of Kurtonina and de Rijke [15], since their notion of bisimulation is not equivalent to a bisimulation for temporal languages preserving simple past and future operators. The reason being, their semantics for the temporal operator "since" and "until" has a universal flavour which is not present in our semantic definition of "reachable from" and "propagate to".

The following theorem is a direct consequence of Lemma 28 and Theorem 17.

▶ **Theorem 29.** *If ρ is a modal bisimulation with converse between two quasi-discrete neighbourhood models $\mathcal{M}_1$ and $\mathcal{M}_2$ with $x_1 \,\rho\, x_2$, then $\mathcal{M}_1, x_1 \models \varphi$ if, and only if, $\mathcal{M}_2, x_2 \models \varphi$ for every formula φ of SLCS.*

5 Related Work

While using logic as a description language for topological properties has a long tradition, for example in the work of Tarski [22], only in recent years there has been a resurgence of spatial interpretations of modal logics. We refer the reader to the survey by Aiello and van Benthem [2], and the different chapters in the Handbook of Spatial Logics [1] for examples of topological, geometric, and other interpretations. While the topologic interpretations allow for a topological bisimulation, the neighbourhood bisimulation we present in this work is more general, since it is defined for a larger class of spaces. However, it is straightforward to show that on topological models (cf. Def. 12), topological bisimulation and neighbourhood bisimulation coincide. A different line of work that is more related to the study of bisimulations is the spatial logic for concurrency [8], which allows for the structural analysis of pi-calculus processes [17].

Our work directly builds on the definitions of SLCS by Ciancia et al. [10]. Besides a model checking algorithm for SLCS, they also propose two extensions to the logic. In the first one, SLCS is extended to incorporate a temporal dimension, which is treated with different operators than the spatial ones, i.e., the temporal operators from computation tree logic. Here, we have instead concentrated solely on the spatial aspects of the language, and leave temporal extensions of our bisimulations as future work. In the second extension, SLCS is equipped with set based modalities, e.g., a modality $\mathcal{G}\varphi$ that states the existence of a *path-connected* set B, such that all elements of B satisfy φ. We intend to examine this type of modality in the future. Recently, Ciancia et al. investigated SLCS with coalgebraic methods [11]. They provide several definitions of bisimulations, both with and without a coalgebraic flavour, on quasi-discrete models, show that they coincide, and present an algorithm and an implementation to minimise a given model with respect to these bisimulations. Furthermore, they prove that on the class of quasi-discrete models, where every node has only finitely many pre- and successors, logical equivalence is a bisimulation. On general models, however, their analysis only considers SLCS without path modalities, i.e., the only spatial modality allowed is *near*. They define a bisimulation, which is similar to definition of neighbourhood bisimulation, and prove that it coincides with logical equivalence induced by an infinitary modal logic.

The logic STREL of Bartocci et al. [4] is another extension to SLCS, where the modalities are defined to be metric with respect to different distance functions. That is, for example, they can express that conditions only hold for paths "up to three steps", and similar properties. Therefore, extending our bisimulations to metric bisimulations in this way is not trivial. In particular, we strongly suspect this would imply using a kind of metric space as the index space. However, in typical settings, it is not desirable for the "metric" to be symmetric. For example, in directed graphs, the distance from x to y may be different from the other way around. Such a situation calls for *quasi-metrics*, which only satisfy the triangle inequality, and that points of distance zero are identical [24].

Neighbourhood semantics of modal logics have been studied quite extensively by now [18]. However, there are subtle differences to the situation of our neighourhood models. For one, the logic we study has different modalities than standard modal logic. In particular, while the *near* modality is equivalent to the diamond-modality of modal logics with neighbourhood semantics, the path-based modalities are more expressive. Furthermore, the spatial interpretation of neighbourhood semantics is only concerned with topological spaces, while we are considering the more general notion of arbitrary neighbourhood spaces.

6 Conclusion

We have presented path preserving bisimulation, a bisimulation on spatial models based on neighbourhood spaces, a generalisation of topological spaces. We have then proven that the truth of formulas of the spatial logic SLCS is preserved between bisimilar points on the models. Using these results, we have shown that SLCS is not strong enough to express certain topological properties, such as separation properties or connectedness. Furthermore, we have compared this bisimulation with more standard approaches on the subset of purely quasi-discrete models proving that it coincides with modal bisimulation with converse.

There are several natural ways to extend this line of work. Up to now, we have only shown that bisimilarity implies the invariance of formulas. However, it is important to investigate whether our bisimulations are matching invariance of formulas exactly, i.e., whether two points that satisfy the same set of formulas are also bisimilar. Here, results of Kurtonina and de Rijke with respect to temporal models might be promising [15], but an adaptation is not straightforward. In particular, they show that the ultrapower construction of first-order models yields models that are suitably saturated to contain witnesses of all necessary types. However, this approach is reliant on the standard translation of modal logic into first-order logic, a result we do not have at our disposal. This is due to the second-order nature of the path modalities, which cannot be reduced to first-order in a similar way as in temporal logic.

It is immediate that for quasi-discrete models, image-finiteness of the edge relation means that the minimal neighbourhood of every point is finite. In this case, the equivalence of points satisfying the same SLCS formulas not using the reachability modality can easily be proven to be a "forward path" preserving bisimulation. But to treat the full logic SLCS, we need an even stronger notion to obtain a class of models where equivalence of formulas is a bisimulation. Even restricting the models such that every point only possesses finitely many successors *and* predecessors is not sufficient. This is due to the fact that *reachable* quantifies over paths that meet the current point, i.e., in a way we can refer to "backwards" paths, but it is not possible to refer to the immediate predecessor of a point. To alleviate this, we could introduce a converse modality to *near*, to distinguish points appropriately. Ciancia et al. [11] achieved such a distinction by employing "strong" variants of the reachability modalities, which allows them to define such a converse modality as an abbreviation.

Regarding the existing extensions of SLCS with set-based modalities, we are interested in studying how far our notion of bisimulations imply the preservation of such modalities, and whether and how we would need to strengthen the definitions. A potentially larger addition would be the investigation of metric variants of SLCS [4], and what kind of metrics or generalised metrics are appropriate in this case.

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A Proofs of Section 3

▶ **Lemma 16** (restated). *Let (Z_η, Z_1, Z_2) be a path preserving bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$, and φ be an SLCS formula that is invariant under neighbourhood bisimulation, i.e., for any $x_1 \in \mathcal{M}_1$ and $x_2 \in \mathcal{M}_2$ with $x_1 Z_\eta x_2$, we have $\mathcal{M}_1, x_1 \models \varphi$ if, and only if, $\mathcal{M}_2, x_2 \models \varphi$. For two paths p and q with $(p, n)\ Z_1\ (q, m)$, we have $\mathcal{M}_1, p, (0, n) \models \varphi$ implies $\mathcal{M}_2, q, (0, m) \models \varphi$. Additionally, if $(q, m) Z_2 (p, n)$ then $\mathcal{M}_2, q, (0, m) \models \varphi$ implies $\mathcal{M}_1, p, (0, n) \models \varphi$.*

Proof. Assume $\mathcal{M}_1, p, (0, n) \models \varphi$ and $(p, n)\ Z_1\ (q, m)$, and let k_q be an arbitrary index such that $0 < k_q < m$. We need to show that $\mathcal{M}_2, q(k_q) \models \varphi$. By the bisimulation property (Def. 15 (4)), we know that there is a k_p such that $0 < k_p < n$ and $p(k_p) Z_\eta q(k_q)$. By the semantics of path intervals, we have $\mathcal{M}_1, p(k_p) \models \varphi$, and since φ is invariant under neighbourhood bisimulation, we get $\mathcal{M}_2, q(k_q) \models \varphi$. Since k_q was arbitrary, we have $\mathcal{M}_2, q, (0, m) \models \varphi$. The other case is similar. ◀

▶ **Theorem 17** (restated). *If (Z_η, Z_1, Z_2) is a path preserving bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$ with $x_1 Z_\eta x_2$, then $\mathcal{M}_1, x_1 \models \varphi$ if, and only if, $\mathcal{M}_2, x_2 \models \varphi$ for every formula φ of SLCS.*

Proof. We proceed by induction on the length of formulas. The induction base and the cases for the Boolean operators are as usual.

So consider $\mathcal{M}_1, x_1 \models \mathcal{N}\varphi$. That is, $x_1 \in \mathcal{C}_1(\{y \mid \mathcal{M}_1, y \models \varphi\})$, which by Def. 2 is equivalent to $x_1 \in \{z \mid \forall N \in \eta_1(z)\colon N \cap \{y \mid \mathcal{M}_1, y \models \varphi\} \neq \emptyset\}$. Hence $\forall N \in \eta_1(x_1)\colon \exists y \in N\colon \mathcal{M}_1, y \models \varphi$. Now choose an arbitrary neighbourhood N_2 of x_2, i.e., $N_2 \in \eta_2(x_2)$. By condition *(frt$_\eta$)* of Def. 13, there is a neighbourhood $N_1 \in \eta_1(x_1)$ such that for all $y_1 \in N_1$, there is a $y_2 \in N_2$ with $y_1 Z_\eta y_2$. In particular, this is the case for the y_1 with $\mathcal{M}_1, y_1 \models \varphi$. Hence, by the induction hypothesis, $\mathcal{M}_2, y_2 \models \varphi$. Since N_2 was arbitrary, we have $\forall N_2 \in \eta_2(x_2)\colon \exists y \in N_2\colon \mathcal{M}_2, y \models \varphi$. That is, $x_2 \in \{z \mid \forall N \in \eta_2(z)\colon N \cap \{y \mid \mathcal{M}_2, y \models \varphi\} \neq \emptyset\} = \mathcal{C}_2(\{y \mid \mathcal{M}_2, y \models \varphi\})$. Hence, $\mathcal{M}_2, x_2 \models \mathcal{N}\varphi$. The other direction is similar.

Now let $\mathcal{M}_1, x_1 \models \varphi\,\mathcal{P}\,\psi$. That is, there is a path p with $p(0) = x_1$ and an n such that $\mathcal{M}_1, p(n) \models \psi$, $\mathcal{M}_1, x_1 \models \varphi$ and $\mathcal{M}_1, p, (0, n) \models \varphi$. Now, by the induction hypothesis, we have $\mathcal{M}_2, x_2 \models \varphi$. Furthermore, by Def. 15, there is a path q on $\mathcal{M}_2$ with $q(0) = x_2$ and m such that $(p, n) Z_1 (q, m)$ and $p(n) Z_\eta q(m)$. Hence, $\mathcal{M}_2, q(m) \models \psi$, and by Lemma 16, we have $\mathcal{M}_2, q, (0, m) \models \varphi$. All in all, $\mathcal{M}_2, x_2 \models \varphi\,\mathcal{P}\,\psi$. The other direction is similar, using Z_2 and the other case of Lemma 16.

The case for $\varphi\,\mathcal{R}\,\psi$ is similar to the preceeding case, using the additional cases in Def. 15 as indicated in the last item. For illustration, we prove the first subcase. So assume $\mathcal{M}_1, x_1 \models \varphi\,\mathcal{R}\,\psi$. Hence, there is a path p on $\mathcal{M}_1$ and an n such that $p(n) = x_1$ and $\mathcal{M}_1, p(0) \models \psi$, $\mathcal{M}_1, x_1 \models \varphi$ and $\mathcal{M}_1, p, (0, n) \models \varphi$. By Def. 15, we then have that there is a path q on $\mathcal{M}_2$ and an m such that $(p, n) Z_1 (q, m)$ and $p(0) Z_\eta q(0)$. By the induction hypothesis, we get $\mathcal{M}_2, q(m) \models \varphi$, $\mathcal{M}_2, q(0) \models \psi$, and then, by Lemma 16, we also have $\mathcal{M}_2, q, (0, m) \models \varphi$. Hence, $\mathcal{M}_2, x_2 \models \varphi\,\mathcal{R}\,\psi$. ◀

B Proofs of Section 4

Proofs in this section rely on definitions of modal bisimulation based on the notion of minimal neighbourhood. This is possible due to the strong relationship between the edge relation and the minimal neighbourhood. In particular, the definition of modal bisimulation can be rewritten in terms of minimal neighbourhood, as *(frt$_f$)* (resp., *(bck$_f$)*) can be rewritten as for every $y_1 \in N_{min}(x_1)$ (resp., $y_2 \in N_{min}(x_2)$) there exists $y_2 \in N_{min}(x_2)$ (resp., $y_1 \in N_{min}(x_1)$) and $y_1 \,\rho\, y_2$. Analogously, the definition of modal bisimulation with converse can be rewritten in terms of minimal neighbourhood, as *(frt$_c$)* (resp., *(bck$_c$)*) can be rewritten as for every $y_1 \in \{y \in X_1 \mid x_1 \in N_{min}(y)\} = \mathcal{C}(x_1)$ (resp., $y_2 \in \{y \in X_2 \mid x_2 \in N_{min}(y)\} = \mathcal{C}(x_2)$) there exists $y_2 \in \mathcal{C}(x_2)$ (resp., $y_1 \in \mathcal{C}(x_1)$) and $y_1 \,\rho\, y_2$.

▶ **Lemma 25** (restated). *On quasi-discrete neighbourhood models, neighbourhood bisimulation and modal bisimulation coincide.*

Proof. Let $\mathcal{M}_1 = (X_1, \eta_1, \nu_1)$ and $\mathcal{M}_2 = (X_2, \eta_2, \nu_2)$ be two quasi-discrete neighbourhood models, and $\rho \subseteq X_1 \times X_2$ a relation between them. We show that ρ is a modal bisimulation iff it is a neighbourhood bisimulation.

($\Rightarrow$) Assume $x_1 \,\rho\, x_2$. Atomic equivalence is trivially true. By *(frt$_f$)* for any $y_1 \in N_{min}(x_1)$ there exists $y_2 \in N_{min}(x_2)$ with $y_1 \,\rho\, y_2$. As $N_{min}(x_2) \subseteq N$ for any $N \in \eta_2(x_2)$, it is always possible to chose $N_{min}(x_1)$ to satisfy the *(frt$_\eta$)* condition. Hence, on quasi-discrete neighbourhood models *(frt$_f$)* implies *(frt$_\eta$)*. The backward direction is analogous.

($\Leftarrow$) Assume $x_1 \,\rho\, x_2$. Atomic equivalence is trivially true. By *(frt$_\eta$)* for $N_{min}(x_2)$ there exists a neighbourhood $N_1 \in \eta_1(x_1)$ such that for every $y_1 \in N_1$ there exists $y_2 \in N_{min}(x_2)$ with $y_1 \,\rho\, y_2$. As $N_{min}(x_1) \subseteq N_1$, it follows that on quasi-discrete neighbourhood models, *(frt$_\eta$)* implies *(frt$_f$)*. The backward direction is analogous. ◀

In order to prove Theorem 26, we first show a stronger result on preservation of paths.

▶ **Lemma 30.** *If ρ is a modal bisimulation between two quasi-discrete neighbourhood models $\mathcal{M}_1$ and $\mathcal{M}_2$ with $x_1 \ \rho\ x_2$, then for every path $p\colon x_1 \rightsquigarrow \infty$ there exists a path $q\colon x_2 \rightsquigarrow \infty$ such that for any $n \in \mathbb{N}$ it holds that $p(n) \ \rho\ q(n)$, and the other way around.*

Proof. We recursively build the path q as follows. First, set $q(0) = x_2$. Second, if $q(k)$ is defined and $p(k) \ \rho\ q(k)$, then by modal bisimulation there exists some $y \in N_{min}(q(k))$ with $p(k+1) \ \rho\ y$, and we set $q(k+1) = y$. By construction we have that $p(n) \ \rho\ q(n)$ for any $n \in \mathbb{N}$. We need to show that q is a continuous function. For quasi-discrete neighbour models this means to show that for any $\{n, n+1\}$ we have that $q[\{n, n+1\}] \subseteq N_{min}(q(n))$, which follows by construction. ◀

▶ **Theorem 26** (restated). *If ρ is a modal bisimulation between two quasi-discrete neighbourhood models $\mathcal{M}_1$ and $\mathcal{M}_2$ with $x_1 \ \rho\ x_2$, then $\mathcal{M}_1, x_1 \models \varphi$ if, and only if, $\mathcal{M}_2, x_2 \models \varphi$ for every formula φ of SLCS without $\mathcal{R}$.*

Proof. We proceed by induction on the length of formulas. The induction base and the cases for the Boolean operators are as usual.

Consider $\mathcal{M}_1, x_1 \models \mathcal{N}\,\varphi$. On quasi-discrete neighbourhood models this means that there exists $x_1' \in N_{min}(x_1)$ such that $\mathcal{M}_1, x_1' \models \varphi$. By *(frt$_f$)*, there exists $x_2' \in N_{min}(x_2)$ such that $x_1' \ \rho\ x_2'$ and, by IH, $\mathcal{M}_2, x_2' \models \varphi$. Hence, $\mathcal{M}_1, x_2 \models \mathcal{N}\,\varphi$. The other direction is similar.

Consider $\mathcal{M}_1, x_1 \models \varphi\,\mathcal{P}\,\psi$. That is, there is a path p and an n such that $p(0) = x_1$ and $\mathcal{M}_1, p(i) \models \varphi$ for all $0 \leq i < n$, and $\mathcal{M}_1, p(n) \models \psi$. By Lemma 30 there exists a path q on $\mathcal{M}_2$ with $q(0) = x_2$, and such that $p(i) \ \rho\ q(i)$ for all $i \in \mathbb{N}$. Then by IH, $\mathcal{M}_2, q(i) \models \varphi$ for all $0 \leq i < n$, and $\mathcal{M}_2, q(n) \models \psi$. Hence, $\mathcal{M}_2, x_2 \models \varphi\,\mathcal{P}\,\psi$ The other direction is similar. ◀

In order to prove Lemma 28, we first show a stronger result on preservation of paths.

▶ **Lemma 31.** *If ρ is a modal bisimulation with converse between two quasi-discrete neighbourhood models $\mathcal{M}_1$ and $\mathcal{M}_2$ with $x_1 \ \rho\ x_2$, then for every path $p\colon y_1 \rightsquigarrow \infty$ with $p(n) = x_1$*

there exists a path $q\colon y_2 \rightsquigarrow \infty$ *with* $q(n) = x_2$ *such that for any* $i \in \mathbb{N}$ *it holds that* $p(i)\ \rho\ q(i)$*, and the other way around.*

Proof. We recursively build the path q as follows. First we set $q(n) = x_2$, and all $q(i)$ values with $i \geq n$ are defined as in Lemma 30. Second, if $q(k)$ with $0 < k \leq n$ is defined and $p(k)\ \rho\ q(k)$, then by modal bisimulation with converse there exists some y with $q(k) \in N_{min}(y)$ and $p(k-1)\ \rho\ y$, and we set $q(k-1) = y$. By construction we have that $p(i)\ \rho\ q(i)$ for any $i \in \mathbb{N}$, and continuity of q is as in Lemma 30. ◀

▶ **Lemma 28** (restated). *On quasi-discrete neighbourhood models, path preserving bisimulation and modal bisimulation with converse coincide.*

Proof. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two quasi-discrete neighbourhood models. To prove the lemma, we show that (1) if (Z_η, Z_1, Z_2) is a path preserving bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$, then Z_η is a modal bisimulation with converse; and (2) if ρ is a modal bisimulation with converse, ρ induces a path preserving bisimulation (ρ, Z_1, Z_2).

(1). Assume $x_1\ Z_\eta\ x_2$. Atomic equivalence is trivially true. By point 2 of Definition 15 for any path $p\colon x_1 \rightsquigarrow \infty$ and $n \neq 0$ there exists $q\colon x_2 \rightsquigarrow \infty$ and m s.t. $p(n)\ Z_\eta\ q(m)$. On quasi-discrete neighbourhood models, if $n = 1$, then $m = 1$ and we have that for any $y_1 \in N_{min}(x_1)$ there exists $y_2 \in N_{min}(x_2)$ s.t. $y_1\ Z_\eta\ y_2$. Hence, Z_η satisfies *(frt$_f$)*. The direction for *(bck$_f$)* is analogous by point 5 of Definition 15, and a similar argument also holds for *(frt$_c$)* and *(bck$_c$)* by points 3 and 6 of Definition 15.

(2). Assume $x_1\,\rho\,x_2$ and let $p\colon x_1 \rightsquigarrow \infty$ be a path starting from x_1. By Lemma 30 there exists a path $q\colon x_2 \rightsquigarrow \infty$ s.t. $p(i)\ \rho\ q(i)$ for all $i \in \mathbb{N}$. Let us set $(p, n)\ Z_1^{p,n}\ (q, n)$ and $Z_2^{q,n}$ as the inverse of $Z_1^{p,n}$. Let $Z_1 = \bigcup_{p \in \mathcal{P}, i>0} Z_1^{p,i}$ and $Z_2 = \bigcup_{q \in \mathcal{Q}, i>0} Z_2^{q,i}$ with $\mathcal{P}$ (resp., $\mathcal{Q}$) the set of paths over $\mathcal{M}_1$ (resp., $\mathcal{M}_2$) starting from bisimilar points. It is immediate that (ρ, Z_1, Z_2) satisfies points 2, 4, 5 and 7 of Definition 15. The cases for points 3 and 6 of Definition 15 is analogous by using paths defined in the proof of Lemma 31. Hence, ρ induces a path preserving bisimulation (ρ, Z_1, Z_2). ◀