Predicates and Quantifiers

Predicates (aka propositional functions)

Propositions (things that are true or false) that contain variables

$P(x):\ x > 0$

P(-2) is false
P(42) is true
P(0) is false
…

- predicates become propositions (true or false) if
  - variables are bound with values from *domain of discourse U*
  - variables are quantified (more in a minute)

The above predicate, we need to state what values x can take, i.e. what is its domain of discourse?

Let U = Z, the set of integers {…,-2,-1,0,1,2,…}

Predicates (aka propositional functions)

$P(x):\ x > 0$

Let U = Z, the set of integers {…,-2,-1,0,1,2,…}

$P(y) \vee \neg P(0)$ ← Not a proposition. Variable y is unbound

Predicates (aka propositional functions)

$R(x, y, z):\ x + y = z$

Let U = Z, the set of integers {…,-2,-1,0,1,2,…}

Put another way: "Let the domain of discourse be the set of all integers"

What is:
- R(2,-1,3) ?
- R(x,3,z) ?
- R(3,6,9) ?

What is a predicate/propositional function?

- A boolean function, i.e. delivers as a result true or false
- isOdd(x), isEven(x), isMarried(x), isWoman(x) …
- isGreaterThan(x,y)
- sumsToOneHundred(a,b,c,d,e)
- In Claire (a nice language)
  - [P(x:integer) : boolean -> x > 3]
- In Claire (a nice language)
  - [R(x:integer,y:integer,z:integer) : boolean -> x + y = z]
- isGoingMad(x)
- hasALife(x)
- oddP(x)

Some Examples

- [P(x:integer) : boolean -> x > 3]
- [Q(x:integer,y:integer) : boolean -> x + y = 0]
- [R(x:integer,y:integer,z:integer) : boolean -> x + y = z]
- For P, Q, and R universe of discourse (domain) is set of integers

Quantifiers (universal)

The universal quantifier asserts that a property holds *for all* values of a variable in a given domain of discourse

$\forall$ ← "for all"

$\forall x\ P(x)$ — "for all x P(x) holds"

But what's the domain of discourse? We must state this!

Could also do this
For all integers, P(x) holds → $\forall x \in Z\ P(x)$

Quantifiers (universal)

$\forall x \in \{1,2,3\}\ P(x)$ ⟷ $P(1) \wedge P(2) \wedge P(3)$

Same thing!

AND

Quantifiers (existential)

The existential quantifier asserts that a property holds *for some* values of a variable in a given domain of discourse

$\exists$ ← "there exists"

$\exists x\ P(x)$ — "there exists a value of x such that P(x) holds"

But what's the domain of discourse? We must state this!

Could also do this
There is an integer value of x such that P(x) holds → $\exists x \in Z\ P(x)$

Quantifiers (existential)

$\exists x \in \{1,2,3\}\ P(x)$ ⟷ $P(1) \vee P(2) \vee P(3)$

Same thing!

OR

Quantifiers

Let the universe of discourse U be the set of real numbers

$P(x,y,z):\ xy = z$

$\forall x \forall y \exists z\ P(x,y,z)$

True or false?

So, we can nest quantifiers: example overleaf

Quantifiers

Nesting: what do these mean?

Let the universe of discourse U be the set of integers

$P(x,y):\ x > y$

$\forall x \forall y\ P(x,y)$

$\forall x \exists y\ P(x,y)$

$\exists x \forall y\ P(x,y)$

$\exists x \exists y\ P(x,y)$

Quantifiers

Nesting: what do these mean?

$P(x, y):\ x < y$

$\forall x \in \{1,2\} \forall y \in \{3,4\}\ P(x, y)$

$P(1,3) \wedge P(1,4) \wedge P(2,3) \wedge P(2,4)$

Quantifiers

Nesting: what do these mean?

$P(x, y):\ x < y$

$\forall x \in \{1,2\} \exists y \in \{3,4\}\ P(x, y)$

$(P(1,3) \vee P(1,4)) \wedge (P(2,3) \vee P(2,4))$

Quantifiers

Nesting: what do these mean?

$P(x, y):\ x < y$

$\exists x \in \{1,2\} \forall y \in \{3,4\}\ P(x, y)$

$(P(1,3) \wedge P(1,4)) \vee (P(2,3) \wedge P(2,4))$

$\exists x \forall y\ P(x, y) \neq \forall x \exists y\ P(x, y)$

Quantifiers

Nesting: what do these mean?

$P(x, y):\ x < y$

$\exists x \in \{1,2\} \exists y \in \{3,4\}\ P(x, y)$

$(P(1,3) \vee P(1,4)) \vee (P(2,3) \vee P(2,4))$

Quantifiers and negation

$\neg \exists x\ P(x) \equiv \forall x\ \neg P(x)$

$\neg \forall x\ P(x) \equiv \exists x\ \neg P(x)$

Some Examples

- [P(x:integer) : boolean -> x > 3]
- [Q(x:integer,y:integer) : boolean -> x + y = 0]
- [R(x:integer,y:integer,z:integer) : boolean -> x + y = z]
- For P, Q, and R universe of discourse (domain) is set of integers

$\forall x P(x)$ This is false

$\exists y \forall x Q(x, y)$ There is no single value of y for this

$\forall x \exists y Q(x, y)$ Yip! Y will equal x!

NOTE: sensitivity of order of quantification

Beware!

$$\forall x \forall y P(x, y) \Leftrightarrow \forall y \forall x P(x, y)$$

That's okay

$$\forall x \exists y P(x, y) \Leftrightarrow \exists y \forall x P(x, y)$$

Bad Karma!

NOTE!

If $\exists y \forall x P(x, y)$ is true (i.e. we found a y)
then $\forall x \exists y P(x, y)$ is also true!

If $\forall x \exists y P(x, y)$ is true
then it does not follow that $\exists y \forall x P(x, y)$

Examples

- P(x): x is a lion
- Q(x): x is fierce
- R(x): x drinks coffee
- Universe of discourse
  - all creatures, great and small

All lions are fierce $\forall x(P(x) \rightarrow Q(x))$

Some lions don't drink coffee $\exists x(P(x) \land \neg R(x))$

Some fierce creatures do not drink coffee $\exists x(Q(x) \land \neg R(x))$

Even more examples

Everyone has a best friend

- B(x,y): x's best friend is y
- Universe of discourse
  - people

$$\forall x \exists y \forall z (B(x, y) \land y \neq z \rightarrow \neg B(x, z))$$

When will these examples stop?!

If somebody is a female and she's a parent then she is someone's mother

- F(x): x is a female
- P(x): x is a parent
- M(x,y): x is the mother of y
- Universe of discourse
  - people

$$\forall x(F(x) \land P(x) \rightarrow \exists y M(x, y))$$

Can I think of this stuff in some concrete way?

$\forall x \forall y P(x, y) \quad U = \{1,2,3,4\}$

```
Okay := true
for x in U while okay
do for y in U while okay
     do okay := P(x,y)
okay
```

Can I think of this stuff in some concrete way?

$\forall x \exists y P(x, y) \qquad U = \{1,2,3,4\}$

```
Okay := true
for x in U while okay
do begin
    okay := false
    for y in U while not(okay)
    do okay := P(x,y)
    end
okay
```

Can I think of this stuff in some concrete way?

$\exists x \forall y P(x, y) \qquad U = \{1,2,3,4\}$

```
Okay := false
for x in U while not(okay)
do begin
    okay := true
    for y in U while okay
    do okay := P(x,y)
    end
okay
```

Can I think of this stuff in some concrete way?

$\exists x \exists y P(x, y) \qquad U = \{1,2,3,4\}$

```
Okay := false
for x in U while not(okay)
do for y in U while not(okay)
    do okay := P(x,y)
okay
```

Non-trivial example: arc-consistency

$$\forall i \in \{1 \ldots n\} \forall j \in \{1..n\} \forall x \in d_i \exists y \in d_j \; consistent(i, x, j, y)$$

"For any pair of variables (i,j), for all values in the domain of variable i there will exist at least one value in the domain of variable j such that we can instantiate variable i to the value x and variable j to the value y and it will be consistent"

$d_1 = \{1,2,3\}$

$d_2 = \{1,2,3\}$

$d_3 = \{1,2,3\}$

$v_1 < v_2$

$v_2 = v_3$

Is this an arc-consistent state?