Set Operations

When sets are equal

$$A = B \leftrightarrow \forall x[x \in A \leftrightarrow x \in B]$$

A equals B iff for all x, x is in A iff x is in B

$$A = B \leftrightarrow \forall x[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$$

or

$$A = B \leftrightarrow A \subseteq B \wedge B \subseteq A$$

... and this is what we do to prove sets equal

Union of two sets

Give me the set of elements, x where x is in A or x is in B

$$A \cup B \Leftrightarrow \{x \mid x \in A \vee x \in B\}$$

Example

$A = \{1,2,3,4,5\}$ $B = \{4,5,6,7,8\}$

$A \cup B = \{1,2,3,4,5,6,7,8\}$

A	B	$A \cup B$
0	0	0
0	1	1
1	0	1
1	1	1

OR

Intersection of two sets

Give me the set of elements, x where x is in A and x is in B

$$A \cap B \Leftrightarrow \{x \mid x \in A \wedge x \in B\}$$

Example

$A = \{1,2,3,4,5\}$ $B = \{4,5,6,7,8\}$

$A \cap B = \{4,5\}$

A	B	$A \cap B$
0	0	0
0	1	0
1	0	0
1	1	1

AND

Intersection of two sets

$$A \cap B = \{\} \rightarrow \text{disjoint}(A, B)$$

Difference of two sets

Give me the set of elements, x where x is in A and x is not in B

$$A - B \Leftrightarrow \{x \mid x \in A \wedge x \notin B\}$$

Example

$A = \{1,2,3,4,5\}$ $B = \{4,5,6,7,8\}$

$A - B = \{1,2,3\}$

$$A - B \equiv A \cap \overline{B}$$

A	B	$A - B$
0	0	0
0	1	0
1	0	1
1	1	0

$A \wedge \neg B$

Symmetric Difference of two sets

Give me the set of elements, x
where x is in A and x is not in B OR
x is in B and x is not in A

$$A \oplus B \Leftrightarrow (A - B) \cup (B - A)$$

Example

$A = \{1,2,3,4,5\}$ $B = \{4,5,6,7,8\}$

$A \oplus B = \{1,2,3,6,7,8\}$

A	B	$A \oplus B$
0	0	0
0	1	1
1	0	1
1	1	0

XOR

Complement of a set

Give me the set of elements, x where x is not in A

$$\overline{A} \Leftrightarrow \{x \mid x \notin A\}$$

Example

$A = \{1,2,3,4,5\}$

$U = \{0,1,2,3,4,5,6,7,8,9,10\}$ U is the "universal set"

$$\overline{A} \Leftrightarrow U - A$$

$$\overline{A} = \{0,6,7,8,9,10\}$$

Not

Cardinality of a Set

In claire

- A = {1,3,5,7}
- B = {2,4,6}
- C = {5,6,7,8}

- |A u B| ?
- |A u C|
- |B u C|
- |A u B u C|

Cardinality of a Set

$$|A \cup B| = |A| + |B| - |A \cap B|$$

The principle of inclusion-exclusion

Set Identities

$A \cup \{\} = A$

$A \cap U = A$

Identity

$A \cup U = U$

$A \cap \{\} = \{\}$

Domination

$A \cup A = A$

$A \cap A = A$

Indempotent

- Think of
  - U as true (universal)
  - {} as false (empty)
  - Union as OR
  - Intersection as AND
  - Complement as NOT

Set Identities

$A \cup B = B \cup A$

$A \cap B = B \cap A$

Commutative

$A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cap C) = (A \cap B) \cap C$

Associative

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Distributive

$\overline{A \cup B} = \overline{A} \cap \overline{B}$

$\overline{A \cap B} = \overline{A} \cup \overline{B}$

De Morgan

Four ways to prove two sets A and B equal

- a membership table
- a containment proof
  - show that A is a subset of B
  - show that B is a subset of A
- set builder notation and logical equivalences
- Venn diagrams

$RTP: \overline{A \cap B} = \overline{A} \cup \overline{B}$

Prove lhs is a subset of rhs

Prove rhs is a subset of lhs

$RTP: \overline{A \cap B} = \overline{A} \cup \overline{B}$

… set builder notation and logical equivalences

$$\{x \mid x \in \overline{A \cap B}\}$$

$$\{x \mid \neg(x \in A \cap B)\}$$

$$\{x \mid \neg(x \in A \wedge x \in B)\}$$

$$\{x \mid \neg x \in A \vee \neg x \in B\}$$

$$\{x \mid x \in \overline{A} \vee x \in \overline{B}\}$$

$$\{x \mid x \in \overline{A} \cup \overline{B}\}$$

Class

$RTP: \overline{A \cap B} = \overline{A} \cup \overline{B}$

prove using membership table

Class

$RTP: \overline{A \cup B} = \overline{A} \cap \overline{B}$

prove using set builder and logical equivalence

$RTP: A \cap (B - A) = \{\}$

Prove using set builder and logical equivalences

$$A \cap (B - A)$$

$$\Leftrightarrow \{x \mid x \in A \wedge x \in (B - A)\}$$

$$\Leftrightarrow \{x \mid x \in A \wedge (x \in B \wedge x \notin A)\}$$

$$\Leftrightarrow \{x \mid x \in A \wedge x \notin A \wedge x \in B\}$$

$$\Leftrightarrow \{x \mid x \in \{\} \wedge x \in B\}$$

$$\Leftrightarrow \{\}$$

Collections of sets

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$