Functions

What's that?

Let A and B be sets
- A function is
  - a mapping from elements of A
  - to elements of B
- and is a subset of A×B
  - i.e. can be defined by a set of tuples!

$$f : A \rightarrow B$$

$$\forall x[x \in A \rightarrow \exists y[y \in B \wedge < x, y > \in f]]$$

Example of a function

Let A be the set of women
- {Andrea,Mary,Betty,Susie,Bervely}

Let B be the set of men
- {Phil, Andy, Ian, Dick, Patrick,Desmond}

f: A -> B (i.e. f maps A to B)
- f(a) : is married to … delivers husband

f = {(Andrea,Patrick),(Beverly,Phil),
(Susie,Dick), (Mary,Ian),(Betty,Andy)}

Example of a function

- Andrea
- Mary
- Betty
- Susie
- Beverly

- Phil
- Andy
- Ian
- Dick
- Patrick
- Desmond

Domain

Codomain

A value in the domain maps to only one value in codomain otherwise it is not a function

F(Andrea) = Patrick
- Patrick is the image of Andrea
- Andrea is the preimage of Patrick


$$f : A \rightarrow B$$

- A is the domain
- B is codomain
- f(x) = y
  - y is image of x
  - x is preimage of y
- There may be more than one preimage of y
  - marriage? Wife of? A man with many wives? Kidding?
- There is only one image of x
  - otherwise not a function
- There may be an element in the codomain with no preimage
  - Desmond ain't married
- Range of f is the set of all images of A
  - the set of all results
  - f(A) = {Patrick,Ian,Phil,Andy,Dick}
    - the image of A

## The image of a set S

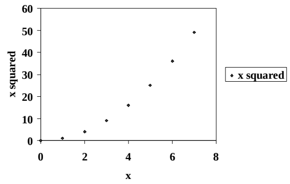
$$f(S) = \{ f(x) \mid x \in S \}$$

F({Andrea,Mary,Beverly})={Ian,Patrick,Phil}

## Another Function

$$f : N \rightarrow N$$
$$f(x) = x^2$$
$$f = \{(0,0), (1,1), (2,4), (3,9), (4,16), \ldots\}$$

A function has a graph!




We can add and multiply functions

… so long as they have the same domains and codomains

## Adding and multiplying functions

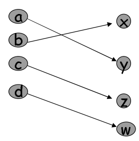
$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$
$$(f_1 f_2)(x) = f_1(x) \times f_2(x)$$
$$f_1(x) = x^2 \qquad f_2(x) = 2x$$
$$(f_1 + f_2)(x) = x^2 + 2x$$
$$(f_1 f_2)(x) = 2x^3$$

Types of functions
- injection
- strictly increasing/decreasing
- surjection
- bijection

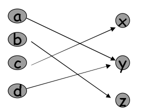
## Injection (aka one-to-one, 1-1)

$$\forall x \forall y [(f(x) = f(y)) \rightarrow x = y]$$
$$\forall x \forall y [(x \neq y) \rightarrow (f(x) \neq f(y))]$$

They say the same thing (note contrapositive!)




injection




not an injection

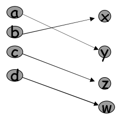
If an injection then preimages are unique

## Injection (aka one-to-one, 1-1)

$$\forall x \forall y [(f(x) = f(y)) \rightarrow x = y]$$

$$\forall x \forall y [(x \neq y) \rightarrow (f(x) \neq f(y))]$$

If f:A→B injective (1-1) then |A|≤|B|




injection

## Strictly increasing/decreasing

- strictly increasing
  - if x< y then f(x) < f(y)
- strictly decreasing
  - if x > y then f(x) > f(y)

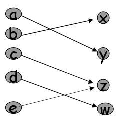
Note: must therefore be 1-1 (I.e. f(x) = f(y) ↔ x=y)

## OnTo/Surjective

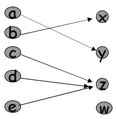
f:A→B is onto/surjective iff for every element b ∈ B there is an element a ∈A such that f(a) = b

$$\forall y \in B \exists x \in A [f(x) = y]$$

Each value in the codomain has a preimage




surjection



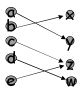

Not a surjection

## OnTo/Surjective

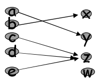
Each value in the codomain has a preimage

***Is this onto/surjective?***

f:A→B where
A = {a,b,c,d}
B = {1,2,3}
f = {(a,3),(b,2),(c,1),(d,3)}



surjection



Not a surjection

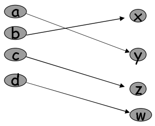
## Bijection (1-1 correspondence)

$$f : A \rightarrow B$$

f is a bijection iff it is both 1-1 and onto

- f(a) = f(b) ↔ a = b (1-1)
- each element of codomain has a preimage (onto)

$$\forall x \forall y (f(x) = f(y) \rightarrow x = y) \wedge \forall b \exists a [b \in B \wedge a \in A \rightarrow f(a) = b]$$




bijection

$$|A| = |B|$$

## Summary

- 1-1/injection/one-to-one
  - f(a) = f(b) ↔ a = b
  - codomain is at least as large as domain
- onto/surjection
  - every element of codomain has a preimage
- bijection
  - 1-1 and onto
  - domain and codomain same size

For the class

Given the following functions state if they are injections (1-1), surjections, or bijections.

$f : Z \to Z$ where $f(x) = x^2$

$g : Z \to \mathrm{E}$ where $f(x) = 2x$
where E is the set of even integers

$h : Z \to \mathrm{Z}$ where $f(x) = \mathrm{x} \div 2$

$p2pc : P \to \mathrm{C}$
where $p2pc(p) =$ postal code of person p

Inverse of a function

$f^{-1}$

Invertable functions

$f : A \to B$
$f(a) = b$

$f^{-1} : B \to A$
$f^{-1}(b) = a$

$f(a) = b \wedge f^{-1}(b) = a$

$\therefore\ f^{-1}(f(a)) = a$

For inverse to exist function must be a bijection

Composition

$f \circ g$

The composition of *f* with *g*

Composition

If g:A→B and f:B→C then (f ∘ g)(x) = f(g(x))

Can only compose functions f and g if the range of g is a subset of the range of f

Composition … an example

Let g be the function from student number numbers to student
Let f be the function from students to postal codes

(f ∘ g)(a) = f(g(a))

delivers the postal code corresponding to a student number

Note:
A is set of student numbers numbers
B is set of students
C is set of postal codes

## Composition … an other example

$f(x) = 2x + 3$

$g(x) = x^2$

$$(f \circ g)(x) = f(g(x)) = f(x^2) = 2x^2 + 3$$

$$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 4x^2 + 12x + 9$$

## For the class

g:A→A where A={a,b,c} and g = {(a,b),(b,c),(c,a)}

f:A→B where B={1,2,3} and f = {(a,3),(b,2),(c,1)}

give compositions f ∘ g and g ∘ f

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## Two important functions

$$floor(x) = \lfloor x \rfloor$$
$$ceiling(x) = \lceil x \rceil$$

## Floor

$floor(x) = \lfloor x \rfloor$

```
[floor(x:float) : integer -> integer!(x)]
//
// The largest integer that is less than or equal to x
// (a) If x >= 0 this amounts to truncation beyond the decimal point
// (b) If x < 0 (negative)
//     then if there is anything after the decimal then
//     truncate and then subtract 1 (i.e. -0.001 becomes -1)
//
```

## Ceiling

$ceiling(x) = \lceil x \rceil$

```
[ceiling(x:float) : integer
 -> if (float!(floor(x)) < x)
      floor(x) + 1
    else
      floor(x)]
//
// The smallest integer that is greater than x
// eg. ceiling(3.1) = 4, ceiling(-3.1) = -3
//
```

## Get a Headache

$$E = \{0,2,4,6,8,\ldots\}$$
$$N = \{0,1,2,3,\ldots\}$$
$$f : N \to E$$
$$f(x) = 2x$$

So, every element in the domain maps to one element in the codomain
Every element in codomain has unique pre-image
The function is bijective
Therefore cardinality of N is same as cardinality of E
What?

*fin*