Relations and their Properties

What is a relation?

- Relationships may exist between elements of a set
  - the set of motorcycles and the set of motorcyclists
  - the set of activities and the resources to do them
  - BA flight numbers and take off times
  - students and subjects
  - resources and costs of using them
  - the set of lecturers and the set of teaching times
  - the set of pairs of compatible values
  - the set of pairs of nogoods
- We now discuss how we can represent and reason with relations

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Binary Relations

A binary relation from A to B is a subset of A x B

- The Cartesian product of two sets, say A and B
- We might represent this as a set of ordered pairs
  - in a pair, first is from A, second is from B

The relation is a set of pairs where first element is from A and second is from B

$$a \in A \wedge b \in B$$

We say ***"a is related to b by R"*** where R is a relation

$$a\, R\, b \Leftrightarrow (a,b) \in R$$

Binary Relations — Example

- Students:
  - Alex, Bea, Cath, Don, Eddie, Fiona
- Subjects:
  - IP1, FP1, AF2
- Let R be the relation of students who passed subjects

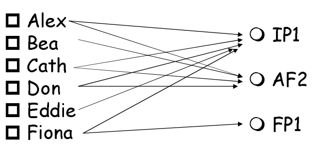
R = {(Alex,IP1),(Alex,AF2),(Bea,AF2),(Cath,AF2),(Cath,IP1),
(Don,AF2),(Don,IP1),(Fiona,IP1),(Eddie,IP1),(Fiona,FP1)}

Order between pairs is insignificant (look at Fiona)
R is a set. Right?
Order within pairs ***is*** significant (a pair (FP2,Fiona)?)

A set of ordered pairs

Binary Relations — Pictorially

R = {(Alex,IP1),(Alex,AF2),(Bea,AF2),(Cath,AF2),(Cath,IP1),
(Don,AF2),(Don,IP1),(Fiona,IP1),(Eddie,IP1),(Fiona,FP1)}



But you could have functional relations

It aint a function!

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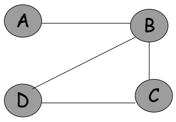
n-ary Relations — A step further

- We have said ***"a relation R"***,
  - meaning ***"a binary relation R"***
- We can have a relation between n sets
  - ***"an n-ary relation R"***
  - ***n = 2, binary*** — ***pairs***
  - ***n = 3, ternary*** — ***triples***
  - ***n = 4, quarternary (?)*** — ***4-tuples***
  - ***n = …, n-ary*** — ***n-tuples***
  - ***n = 1, unary (!)*** — ***singletons***

A set of ordered n-tuples

n-ary Relations — An example



Assume A, B, C, D are to be given values from the set {1,2,3}. If there is an edge between vertices then they must take different values

A 4-tuple (h,i,j,k) corresponds to a consistent assignment A = h, B = i, C = j, D = k

R = {(1,2,1,3),(1,2,3,1),(1,3,2,1),(1,3,1,2),
(2,1,2,3),(2,1,3,2),(2,3,1,2),(2,3,2,1),
(3,1,2,3),(3,1,3,2),(3,2,1,3),(3,2,3,1)}

Functions as Relations

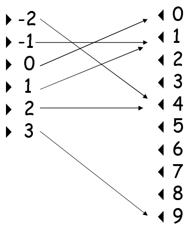
We can represent a function extensionally/explicitly by listing for each value in the domain (pre-image) its value in the co-domain (its image).

That is we can represent the function as a set of pairs, i.e. as a binary relation

Functions as Relations — Example

- F: $A \to B$
  - where $f(x) = x^2$
  - A = {-2,-1,0,1,2,3} and B = {0,1,2,3,4,5,6,7,8,9}

R = {(-2,4),(-1,1),(0,0),(1,1),(2,4),(3,9)}



R	0	1	2	3	4	5	6	7	8	9
−2	0	0	0	0	1	0	0	0	0	0
−1	0	1	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0
2	0	0	0	0	1	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	1

Relation, divisibility — Example

A = {1,2,3,4}
R = {(a,b) | a divides b}

R is the relation "a divides b"
This is defined over the above set *A*

R = {(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)}

R	1	2	3	4
1	1	1	1	1
2	0	1	0	1
3	0	0	1	0
4	0	0	0	1

NOTE: an n-ary relation can be represented using an n-dimensional array

Relations — Example

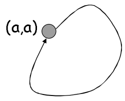
How many relations are there on a set of n elements?

- each relation R_i is a subset of $\{(a,b) \mid a \in A \wedge b \in B\}$
  - R_i is a subset of the Cartesian product of *A* and B
- when we have a relation on a single set, this is just *A* = B
  - R_i is then a subset of $\{(x,y) \mid x \in A \wedge y \in A\}$
- the cardinality of $A \times A$ is $|A \times A| = n^2$
- there are 2^n subsets of a set of size n
- if the set of tuples to choose from is of size n^2
  - then there are $2^{n\hat{}2}$ possible subsets
- there are $2^{(n\hat{}2)}$ possible relations

Properties of Relations — Definitions

- Reflexive
  - if a is in A then (a,a) is in R
- Symmetric
  - if (a,b) is in R and a ≠ b then (b,a) is in R
- Antisymmetric
  - if (a,b) is in R and a ≠ b then (b,a) is not in R
- Transitive
  - if (a,b) is in R and (b,c) is in R then (a,c) is in R

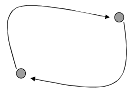
Properties of Relations — Reflexive



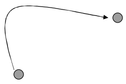
- Reflexive
  - if a is in A then (a,a) is in R
- Example: a divides b i.e. a|b
  - R = {(a,b)| a ∈ A, b ∈ B, a|b}

Properties of Relations — Symmetric

- Symmetric
  - if (a,b) is in R and a ≠ b
  - then (b,a) is in R
- Example: a is married to b



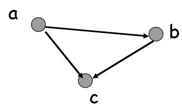
Properties of Relations — Antisymmetric



- Antisymmetric
  - if (a,b) is in R and a ≠ b
  - then (b,a) is not in R
- Example: a divides b i.e. a|b
  - R = {(a,b)| a in A, b in B, a|b}

Properties of Relations — Transitive

- Transitive
  - if (a,b) is in R and (b,c) is in R
  - then (a,c) is in R



- Example: a is less than b i.e. a < b
  - a and b are positive integers
  - R = {(a,b)| a < b}

Combining Relations — Using Set Operations

$$A = \{1,2,3\} \qquad B = \{1,2,3,4\}$$

$$R_1 = \{(1,1),(2,2),(3,3)\}$$

$$R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$$

$$R_1 \cup R_2 = \{(1,1),(1,2),(2,2),(1,3),(1,4),(3,3)\}$$

$$R_1 \cap R_2 = \{(1,1)\}$$

$$R_1 - R_2 = \{(2,2),(3,3)\}$$

$$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$$

Composite Relations — Definition

- Let R be a relation from set A to set B
- Let S be a relation from set B to set C
- The composite of R and S is a relation from set A to set C
  - and is a set of ordered pairs (a,c) such that
  - there exists an (a,b) in R and an (b,c) in S
- The composite of R and S is denoted as SoR

The composite of R with S

Composite Relations — Example

Composite Relations — So?

Assume we have a relation R of people to motorcycles they own.
A person could have more that one motorcycle.
R is a set of ordered pairs {(Patrick,RGV),(Denis,Buell),(Stan,ElectraGlide),(Denis,Hayabusa),(Gordon,Bandit),(Gordon,R6)}

We could have a relation S of motorcycles to top speed
{(RGV,130),(Buell,126),(ElectraGlide,110),(Hayabusa,182),(Bandit,140),(R6,155)}

SoR is then the relation of people to possible top speeds
{(Patrick,130),(Denis,126),(Denis,182),(Stan,110),(Gordon,140),(Gordon,155)}

Composite of a Relation with itself

Let R be a relation on the set A. The powers R^n are defined inductively as follows

Composite of a Relation with itself

A Transitive Relation

A relation R on a set A is transitive iff R^n is a subset of R for n = 1,2,3,...

A Transitive Relation

R is transitive iff $R^n \subseteq R$ for n = 1,2,3,...

Proof

A = {1,2,3,4}

R = {(a,b) | a divides b}

Example

R = {(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)}

RoR = {(1,1),(1,2),(1,3),(1,4),(2,4),(3,3),(4,4)}

RoR is a subset of R, therefore transitive

Obvious! If a divides b and b divides c then a divides c!

Exercise

Let R be a relation on people such that (a,b) is "a is a parent of b"

Let S be a relation on people such that (a,b) is "a is a sibling of b"

- What is SoR, RoS, RoR?
  - SoR composes R with S
    - (a,c) is in SoR if there exists
      - (a,b) in R and a (b,c) in S
      - "a is parent of b" and "b is sibling of c
    - SoR is in R!
  - RoS composes S with R
    - "a is sibling of b" and "b is a parent of c"
    - therefore (a,c) is "a is an aunt/uncle of c"

Exercise

List the 16 relations on the set A = {0,1}

The lecture is over