***Relations: representation and closures***

We can represent a (binary) relation as a 0/1 matrix

$$A = \{a_1, a_2, ..., a_m\}$$
$$B = \{b_1, b_2, ..., b_n\}$$

$M$ is an m by n connection matrix for the relation R

$M_{i,j} = 1 \rightarrow (a_i, b_j) \in R$

$M_{i,j} = 0 \rightarrow (a_i, b_j) \notin R$

Example of a relation as a 0/1 matrix

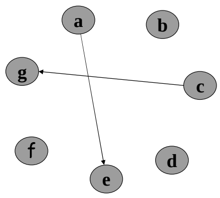
$A = \{a, b, c\}$

$B = \{e, f, g, h\}$

$R = \{(a, e), (c, g)\}$

| $R$ | $e$ | $f$ | $g$ | $h$ |
|---|---|---|---|---|
| $a$ | 1 | 0 | 0 | 0 |
| $b$ | 0 | 0 | 0 | 0 |
| $c$ | 0 | 0 | 1 | 0 |

The digraph of a relation




| $R$ | $e$ | $f$ | $g$ | $h$ |
|---|---|---|---|---|
| $a$ | 1 | 0 | 0 | 0 |
| $b$ | 0 | 0 | 0 | 0 |
| $c$ | 0 | 0 | 1 | 0 |

Properties of relations as a 0/1 matrix

Reflexive

| | | | |
|---|---|---|---|
| 1 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 |

Symmetric

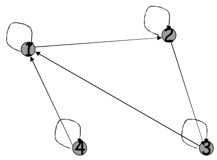
| | | | |
|---|---|---|---|
| 1 | 1 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 0 | 1 | 1 | 0 |
| 0 | 0 | 0 | 1 |

Antisymmetric

| | | | |
|---|---|---|---|
| 1 | 0 | 1 | 1 |
| 1 | 0 | 1 | 0 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 |

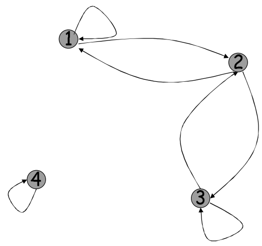
## Reflexive

$$\begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{matrix}$$



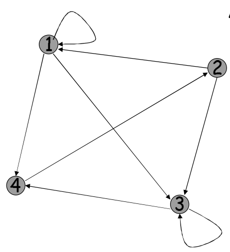

## Symmetric

$$\begin{matrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$




## Antisymmetric

$$\begin{matrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{matrix}$$




## Join and Meets

$$\text{Join}: M_{R_1} \vee M_{R_2}$$

$$m_{i,j} = 1 \vee n_{i,j} = 1 \rightarrow p_{i,j} = 1$$

$$M_{R_1} \vee M_{R_2} \Leftrightarrow R_1 \cup R_2$$

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$$\text{Meets}: M_{R_1} \wedge M_{R_2}$$

$$m_{i,j} = 1 \wedge n_{i,j} = 1 \rightarrow p_{i,j} = 1$$

$$M_{R_1} \wedge M_{R_2} \Leftrightarrow R_1 \cap R_2$$

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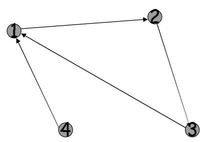
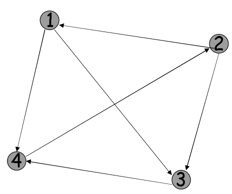
$$M_{R_1} = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix}$$

$$M_{R_2} = \begin{matrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{matrix}$$

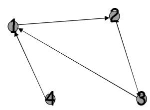


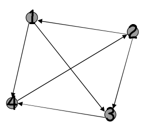




## join

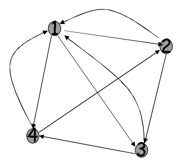
$$M_{R_1} = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix}$$

$$M_{R_2} = \begin{matrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{matrix}$$



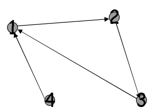



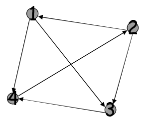

$$M_{R_1} \vee M_{R_2} = \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{matrix}$$



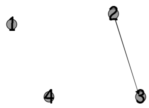
meets

$$M_{R_1} = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix} \qquad M_{R_2} = \begin{matrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{matrix}$$




$$M_{R_1} \wedge M_{R_2} = \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$



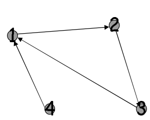

Compose

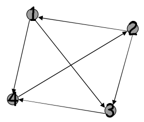
$$(a_i, b_k) \in R \wedge (b_k, c_j) \in S \rightarrow (a_i, c_j) \in S \circ R$$

$$M_{S \circ R} = M_R \otimes M_S \quad \text{(i.e. boolean product)}$$

compose

$$M_{R_1} = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix} \qquad M_{R_2} = \begin{matrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{matrix}$$




$$R_2 \circ R_1 = M_{R_1} \otimes M_{R_2} = \begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{matrix}$$

But what does compose mean here?

What is the intuition behind all of these matrix operations?

Get real?

Closures

If we have a relation R then the closure of R with respect to some property P is the relation S where S is R plus the minimum number of tuples that ensures property P

P could be reflexive, symmetric, transitive

Reflexive Closure

- R = {(1,1),(1,2),(2,1),(3,2)} on the set A = {1,2,3}
- The reflexive closure of R is
  - S = {(1,2),(2,2),(3,3),(1,2),(2,1),(3,2)}
  - we add {(2,2),(3,3)} to R to get S

## Example of Reflexive Closure

## Example of Symmetric Closure

## Transitive Closure

We need to add the minimum number of tuples to R, giving us S, such that if (a,b) is in S and (b,c) is in S, then (a,c) is in S.

## Example of Transitive Closure

R = {(1,3),(1,4),(2,1),(3,2)} on the set A = {1,2,3,4}
What is the transitive closure?

- we have (1,3) and (3,2) so add (1,2)
- we have (3,2) and (2,1) so add (3,1)
- we have (2,1) and (1,3) so add (2,3)
- we have (2,1) and (1,4) so add (2,4)

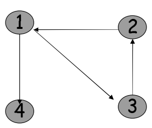
S = {(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1) ,(3,2)}

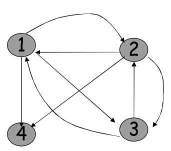
Are we done?

## Example of Transitive Closure

R = {(1,3),(1,4),(2,1),(3,2)}

S = {(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2)}



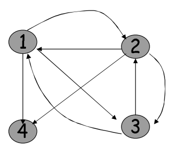




Are we done?

## Example of Transitive Closure

S = {(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2)}

- We have (3,1) and (1,4) but not (1,4)
- We have (1,2) and (2,1) but not (1,1)
- We have ...




Are we done?

## Transitive Closure

## Transitive Closure

$R^2$ contains paths of length 2
$R^3$ contains paths of length 3
$R^4$ contains paths of length 4

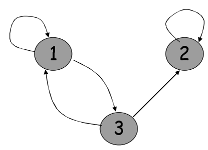
$R \cup R^2 \cup R^3 \cup R^4$
Contains paths of length 1, 2, 3 and 4

$R^* = R \cup R^2 \cup R^3 \cup R^4 \ldots \cup R^n$
Contains paths of length 1, 2, 3, …, n

## Example of Transitive Closure

$$M_R = \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{matrix}$$

$$R = \{(1,1),(1,3),(2,2),(3,1),(3,2)\}$$




## Example of Transitive Closure

$$M_{R^*} = M_R \vee M_R^{\ 2} \vee M_R^{\ 3}$$

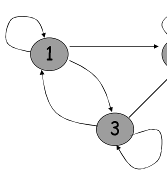
$$M_R = \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{matrix}$$

$$M_R^{\ 2} = M_R \times M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$
\begin{aligned}
M_R{}^3 &= M_R{}^2 \times M_R \\
&= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
M_{R^*} &= M_R \vee M_R{}^2 \vee M_R{}^3 \\
&= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
M_{R^*} &= M_R \vee M_R{}^2 \vee M_R{}^3 \\
&= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}
\end{aligned}
$$




So?