Countability

The cardinality of the set A is equal to the cardinality of a set B if there exists a bijection from A to B

- cardinality?
- bijection?
  - injection
  - surjection

If a set has the same cardinality as a subset of the natural numbers N, then we say is is ***countable***

- Natural numbers N?

If |A| = |N| the set A is ***countably infinite***

Countability implies that there is a listing of the elements of the set (i.e. the first one, the 100th, etc)

If there is an injection from A to B then |A| ≤ |B|

$$\forall x \in A\, \forall y \in B [x \neq y \rightarrow f(x) \neq f(y)]$$

The set of even numbers E is countably infinite

Let f(x) = 2x

There is a bijection from N to E

The set of C programs is countably infinite

- a C program is a string of characters over a given alphabet
- we can order these strings lexicographically
- if a program fails to compile delete it
- we now have an ordered listing of all C programs

This implies a bijection from N to the list of C programs

Therefore C programs are countably infinite

The set of real numbers between 0 and 1 is uncountable

Sketch: We will assume that it is countably infinite and then show that this is absurd.

Assume we can list all the reals between 0 and 1 in a table as follows

$$
\begin{aligned}
r_1 &= d_{1,1}d_{1,2}d_{1,3}d_{1,4}\cdots \\
r_2 &= d_{2,1}d_{2,2}d_{2,3}d_{2,4}\cdots \\
r_3 &= d_{3,1}d_{3,2}d_{3,3}d_{3,4}\cdots \\
r_4 &= d_{4,1}d_{4,2}d_{4,3}d_{4,4}\cdots \\
&\vdots
\end{aligned}
$$

The set of real numbers between 0 and 1 is uncountable

We can now produce a new number that is not in our table

$$
\begin{aligned}
r_1 &= \boldsymbol{\delta_{1,1}}d_{1,2}d_{1,3}d_{1,4}\cdots \\
r_2 &= d_{2,1}\boldsymbol{\delta_{2,2}}d_{2,3}d_{2,4}\cdots \\
r_3 &= d_{3,1}d_{3,2}\boldsymbol{\delta_{3,3}}d_{3,4}\cdots \\
r_4 &= d_{4,1}d_{4,2}d_{4,3}\boldsymbol{\delta_{4,4}}\cdots \\
&\vdots
\end{aligned}
$$

$$x = \delta_{1,1}\delta_{2,2}\delta_{3,3}\delta_{4,4}\cdots$$

Where $\delta_{i,i} \neq d_{i,i}$

There are uncomputable numbers

A number between 0 and 1 is *computable* if there is a C program which when given the input i produces the i^{th} digit of the decimal expansion of that number

Theorem:
There exists a number x between 0 and 1 that is not computable

Proof:
There does not exist a program that will compute it, because the real numbers between 0 and 1 are *uncountable* and the C programs are *countable*, so there are more reals between 0 and 1 than there are C programs.

Our first proof of the limits of computation