JOURNAL OF COMBINATORIAL THEORY (A) 16, 230–240 (1974)

Some Results on Construction of Orthogonal Latin Squares by the Method of Sum Composition

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Communicated by Marshall Hall, Jr.

Received July 3, 1972

A method of sum composition for construction of orthogona Latin squares was introduced by A. Hedayat and E. Seiden [1]. In this paper we exhibit procedures for constructing a pair of orthogonal Latin squares of size $p^\alpha + 4$ for primes of the form $4m + 1$ or $p \equiv 1, 2, 4 \bmod 7$. We also show that for any $p > 2n$ and n even one can construct an orthogonal pair of Latin squares of size $p^\alpha + n$ using the method of sum composition. We observe that the restriction $xy = 1$ used by Hedayat and Seiden is sometimes necessary.

1. INTRODUCTION

DEFINITION. A transversal of a Latin square of order n is a collection of n cells whose entries exhaust the set of distinct elements of the Latin square and such that no two cells belong to the same row or the same column.

Two transversals are called parallel if they have no elements in common.

Hedayat and Seiden [1] introduced the method of sum composition of Latin squares which can be described as follows: Let L_1, L_2 be two Latin squares of order n_1 and n_2 on disjoint sets of elements $\{a_1, a_2, ..., a_{n_1}\}$ and $\{b_1, b_2, ..., b_{n_2}\}$, $n_1 \geqslant n_2$, and let L_1 have at least n_2 parallel transversals. Select arbitrarily n_2 parallel transversals from L_1 and name them $1, 2, ..., n_2$; in a $n_1 + n_2$ size square fill the $n_1 \times n_1$ upper left corner with L_1 and the $n_2 \times n_2$ lower right corner with L_2. Fill the cells $(i, n_1 + k)$, $k = 1, 2, ..., n_2$, with that element of transversal k which appears in row i, $i = 1, 2, \ldots, n_1$; similarly fill the cells $(n_1 + k, j)$, $k = 1, 2, \ldots, n_2$, with that element of transversal k which appears in column j, $j = 1, 2, \ldots, n_1$. Finally substitute b_k for the n_1 elements of transversal k, $k = 1, 2, \ldots, n_2$.

The resulting $n_1 + n_2$ square matrix L is easily seen to be a Latin square.

The procedure just described of filling the first n_1 entries of column (row) $n_1 + k$ is called horizontal (vertical) projection of transversal k on column (row) $n_1 + k$.

Henceforth we shall use the symbol $O(n, 2)$ for a set of two orthogonal Latin squares of order n.

Under certain conditions it is possible to use the method of sum composition to obtain $O(n, 2)$ sets from known $O(n_1, 2)$ and $O(n_2, 2)$ sets, $n = n_1 + n_2$.

Let $\{A_1, A_2\}$ be a $O(n_1, 2)$ set on the set of elements of $A = \{a_1, a_2, \ldots, a_{n_1}\}$ with at least $2n_2$ common parallel transversals, and $\{B_1, B_2\}$ a $O(n_2, 2)$ set on the set of elements of $B = \{b_1, b_2, \ldots, b_{n_2}\}$, $A \cap B = \varnothing$.

Select $2n_2$ common parallel transversals from the first set and use half of them to compose A_1 and B_1 to obtain a Latin square L_1 of order $n_1 + n_2 = n$; use the remainder n_2 transversals to compose A_2 and B_2 to obtain a Latin square L_2 of order n.

It is obvious from the construction that upon superimposition of L_1 on L_2 the elements of $A \times B$ and $B \times A$ will appear along the $2n_2$ transversals in the $n_1 \times n_1$ upper left corner; the elements of $B \times B$ will appear in the $n_2 \times n_2$ lower right corner, since B_1 and B_2 are orthogonal. However some of the elements of $A \times A$ will be missing, but by properly choosing the $2n_2$ transversals and the order of projection we may achieve that the pairs (a_i, a_k) lost by substituting elements of B in transversals of A_1 and A_2 be recovered on projection.

In conclusion we wish to remark that introducing the symbol μ for $(x - 1)/(y - 1)$ reduced the expression for K_h and K_v to a form analogous to that obtained by Hedayat and Seiden due to the assumption $xy = 1$. This helped to realize that this assumption is in fact necessary in case $\sum s_i \neq \sum t_i$. It also helped to find procedures for construction of a pair of orthogonal Latin squares of size $p^\alpha + 4$ for primes of the form $4m + 1$ or congruent to 1, 2, 4 mod 7 and of size $p^\alpha + n$ for any $p > 2n$, n even in case the assumption $xy = 1$ does not hold.

2. Construction of Some $O(n, 2)$ Sets by the Method of Sum Composition

Let $n_1 = p^\alpha$ be a power of a prime p and denote by $A(x)$ a Latin square of order n_1 whose entry in the (i, j) cell is $ix + j \in \mathrm{GF}(n_1)$, $x \neq 0$. Consider two orthogonal Latin squares $A_1 = A(x)$, $A_2 = A(y)$, $x, y \in GF(n_1)$, $x \neq y, \{x, y\} \cap \{0, 1\} = \varnothing$. We can exhibit n_1 common transversals of A_1 and A_2 using the square $A(1)$ whose entries in the cell (i, j) are $i + j$. Let us name the transversal for which $i + j = k$ for any $k \in GF(n_1)$ the transversal k. Since $n_1 \geqslant 2n_2$ we can choose $2n_2$ parallel transversals and partition them into two sets each of size n_2. Let $S = \{s_1, s_2, ..., s_{n_2}\}$ and $T = \{t_1, t_2, ..., t_{n_2}\}$ be two sets of transversals used in the projection process to obtain L_1 and L_2, respectively, as described previously. The problem is to choose these transversals in such a way that the $2n_1n_2$ pairs lost by replacing the entries of the corresponding cells by the elements of the Latin squares of order n_2 are recovered by the projection process. The missing pairs are of the form $(ix + j, iy + j)$, $i + j \in S \cup T$, which correspond to the entries in the $2n_2$ transversals used in the compositions.

If transversal s of $A(x)$ is projected horizontally on the same column as transversal t of $A(y)$, on superimposition we will obtain along that column the n_1 pairs

$$(ax + b, ay + c), \qquad a + b = s, \qquad a + c = t.$$

If those pairs are to be some of the lost ones we must have:

$$ix + j = ax + b, \qquad a + b = s \in S, \qquad a + c = t \in T,$$
$$iy + j = ay + c, \qquad i + j = k \in S \cup T,$$

or

$$i(x - 1) + k = a(x - 1) + s,$$
$$i(y - 1) + k = a(y - 1) + t.$$

Eliminating i we obtain

$$k(y - x) = s(y - 1) - t(x - 1)$$

or

$$k(y - x) = s(y - x) + (s - t)(x - 1).$$

Making $(x - 1)/(y - x) = \mu$ we finally get

$$k = (1 + \mu)s - \mu t;$$

that is, by projecting horizontally transversal s of $A(x)$ on the same column as transversal t of $A(y)$ we obtain on superimposition the n_1 pairs

$$(ix + j, iy + j), \qquad i + j = (1 + \mu)s - \mu t.$$

Similarly, if transversals s and t of $A(x)$, $A(y)$ are projected vertically on the same row, we will obtain along that row the n_1 pairs

$$(ax + b, cy + b), \qquad a + b = s, \qquad c + b = t.$$

If those pairs are to be some of the lost ones we must have

$$ix + j = ax + b, \qquad a + b = s \in S, \qquad c + b = t \in T,$$
$$iy + j = cx + b, \qquad i + j = k \in S \cup T,$$

or

$$i(x - 1) + k = a(x - 1) + s,$$
$$i(y - 1) + k = c(y - 1) + t.$$

Eliminating i we obtain

$$k(y - x) = (x - 1)(y - 1)(a - c) + s(y - 1) - t(x - 1).$$

Since $a - c = s - t$, we get

$$k(y - x) = s(y - x) + (s - t)(x - 1)y$$

and finally

$$k = (1 + y\mu)s - y\mu t;$$

that is, by projecting vertically transversal s of $A(x)$ on the same row as transversal t of $A(y)$ we obtain on superimposition the n_1 pairs

$$(ix + j, iy + j), \qquad i + j = (1 + y\mu)s - y\mu t.$$

From now on we will use the following functions on $S \times T$:

$$K_h(s, t) = (1 + \mu)s - \mu t,$$
$$K_v(s, t) = (1 + y\mu)s - y\mu t.$$

THEOREM 1. *If p is a prime of the form $p = 4m + 1$, $m > 1$, then it is possible to compose $O(p^\alpha, 2)$ based on* GF(p^α) *with $O(4, 2)$ to obtain a $O(p^\alpha + 4, 2)$.*

Proof. Consider the pattern

$$s_{i+1} = K_h(s_i, t_i), \qquad i = 1, 2, 3, \qquad s_1 = K_h(s_4, t_4),$$
$$t_{i-1} = K_v(s_i, t_i), \qquad i = 2, 3, 4, \qquad t_4 = K_v(s_1, t_1);$$

that is

$$
\begin{aligned}
s_2 &= (1+\mu)s_1 - \mu t_1, & t_4 &= (1+y\mu)s_1 - y\mu t_1, \\
s_3 &= (1+\mu)s_2 - \mu t_2, & t_1 &= (1+y\mu)s_2 - y\mu t_2, \\
s_4 &= (1+\mu)s_3 - \mu t_3, & t_2 &= (1+y\mu)s_3 - y\mu t_3, \\
s_1 &= (1+\mu)s_4 - \mu t_4, & t_3 &= (1+y\mu)s_4 - y\mu t_4.
\end{aligned}
$$

Solving this linear system in terms of s_1 and t_1, we obtain as a solution

$$
\begin{aligned}
s_2 &= (1+\mu)\, s_1 - \mu t_1, \\
s_3 &= (1+\mu)\left[1+\mu - \frac{1}{y}(1+y\mu)\right] s_1 \\
&\quad - \left[\mu(1-\mu) - \frac{1}{y}[\mu(1+y\mu)+1]\right] t_1, \\
s_4 &= [\mu(1+y\mu)+1]\frac{1}{1+\mu} s_1 - \frac{y\mu^2}{1+\mu} t_1, \\
t_2 &= \left[(1+y\mu)(1+\mu)\frac{1}{y\mu}\right] s_1 - [\mu(1+y\mu)+1]\frac{1}{y\mu} t_1, \\
t_3 &= \left[(1+y\mu)\frac{1}{1+\mu}[\mu(1+y\mu)+1] - y\mu(1+y\mu)\right] s_1 \\
&\quad - \left[(1+y\mu)\, y\mu^2 \frac{1}{1+\mu} - y^2\mu^2\right] t_1, \\
t_4 &= (1+y\mu)\, s_1 - y\mu t_1.
\end{aligned}
$$

It is easy to check that the requirement that the solutions exhaust the set $S \cup T$, equivalently that all the lost $2n_1n_2$ be recovered by the rows and columns of the projections, reduces the rank of the system to at most four. It is seen that if $s_1 \neq t_1$ then the following equation must hold:

$$(1+\mu)^3 - (1+\mu)^2\, y\mu + (1+\mu)\, y^2\mu^2 - y^3\mu^3 = 0.$$

Dividing by $y^3\mu^3$ and making $(1+\mu)/y\mu = \lambda$ we obtain

$$\lambda^3 - \lambda^2 + \lambda = 1 = 0 \qquad \text{or} \qquad (\lambda - 1)(\lambda^2 + 1) = 0.$$

$\lambda = 1$ would give $s_3 = s_1$, therefore we must have $\lambda^2 + 1 = 0$, that is, -1 has to be a quadratic residue in $\mathrm{GF}(p^\alpha)$, which is possible only if p is of the form $p = 4m + 1$.

Calling $i^2 = -1$, the condition becomes

$$y(1 \pm i(1 - x)) = 1,$$

which is satisfied by the pair $x = 2$, $y = (1 \pm i)/2$. Using $s_1 = 0$, $t_1 = 1$ we obtain as solution of the system

$$s_2 = \frac{3 \pm i}{5}, \qquad t_2 = \frac{3 \mp 4i}{5},$$

$$s_3 = \frac{4 \mp 2i}{5}, \qquad t_3 = \frac{-1 \mp 2i}{5},$$

$$s_4 = \frac{1 \mp 3i}{5}, \qquad t_4 = \frac{1 \pm 2i}{5}.$$

To conclude the proof of the theorem we have to show that the solutions exhibited here are distinct for all values of $p = 4m + 1$, $m > 1$. By considering the 28 differences it is easy to see that both values for s_i and t_i, $i = 2, 3, 4$, are admissible.

To illustrate the theorem we shall compose $O(17, 2)$ with $O(4, 2)$ to obtain $O(21, 2)$. We shall use $y = (1 + i)/2$ with $i = -4$ and $s_1 = 0$, $t_1 = 1$. Then $s_2 = 10$, $s_3 = 16$, $s_4 = 6$, $t_2 = 14$, $t_3 = 15$, $t_4 = 2$. We shall obtain two orthogonal Latin squares of order 21 substituting in $A(2)$ for the entries having cells of $A(1)$ 0, 10, 16, and 6 A, B, C, and D, respectively. In $A(7)$ we shall substitute A, B, C, and D in the places corresponding to 1, 14, 15, and 2 in $A(1)$. The resulting orthogonal squares of size 21 will have the form:

THEOREM 2. *If* $p \equiv 1, 2, 4 \pmod 7$, $p \geqslant 11$ *it is possible to compose* $O(p^\alpha, 2)$ *based on* $\mathrm{GF}(p^\alpha)$ *with* $O(4, 2)$ *to obtain a* $O(p^\alpha + 4, 2)$.

Proof. Consider the pattern

$$s_1 = K_h(s_2, t_2), \qquad t_1 = K_v(s_2, t_2),$$

$$s_2 = K_h(s_3, t_3), \qquad t_2 = K_v(s_3, t_3),$$

$$s_3 = K_h(s_4, t_4), \qquad t_3 = K_v(s_1, t_4),$$

$$s_4 = K_h(s_1, t_1), \qquad t_4 = K_v(s_4, t_1).$$

Using the same method as in Theorem 1 we may solve this system of equations in terms of s_2 and t_2. Imposing the condition that $s_2 \neq t_2$ we shall conclude that the following equation must hold.

$$1 - \mu(y - 1) - \mu^2(y - 1)^2(\mu^2 y + \mu y - 1) = 0.$$

A	1	2	3	4	5	D	7	8	9	B	11	12	13	14	15	C	0	10	16	6
2	3	4	5	6	D	8	9	10	B	12	13	14	15	16	C	A	1	11	0	7
4	5	6	7	D	9	10	11	B	13	14	15	16	0	C	A	3	2	12	1	8
6	7	8	D	10	11	12	B	14	15	16	0	1	C	A	4	5	3	13	2	9
8	9	D	11	12	13	B	15	16	0	1	2	C	A	5	6	7	4	14	3	10
10	D	12	13	14	B	16	0	1	2	3	C	A	6	7	8	9	5	15	4	11
D	13	14	15	B	0	1	2	3	4	C	A	7	8	9	10	11	6	16	5	12
14	15	16	B	1	2	3	4	5	C	A	8	9	10	11	12	D	7	0	6	13
16	0	B	2	3	4	5	6	C	A	9	10	11	12	13	D	15	8	1	7	14
1	B	3	4	5	6	7	C	A	10	11	12	13	14	D	16	0	9	2	8	15
B	4	5	6	7	8	C	A	11	12	13	14	15	D	0	1	2	10	3	9	16
5	6	7	8	9	C	A	12	13	14	15	16	D	1	2	3	B	11	4	10	0
7	8	9	10	C	A	13	14	15	16	0	D	2	3	4	B	6	12	5	11	1
9	10	11	C	A	14	15	16	0	1	D	3	4	5	B	7	8	13	6	12	2
11	12	C	A	15	16	0	1	2	D	4	5	6	B	8	9	10	14	7	13	3
13	C	A	16	0	1	2	3	D	5	6	7	B	9	10	11	12	15	8	14	4
C	A	0	1	2	3	4	D	6	7	8	B	10	11	12	13	14	16	9	15	5
0	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	A	B	C	D
3	2	1	0	16	15	14	13	12	11	10	9	8	7	6	5	4	B	A	D	C
15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0	16	C	D	A	B
12	11	10	9	8	7	6	5	4	3	2	1	0	16	15	14	13	D	C	B	A

0	A	D	3	4	5	6	7	8	9	10	11	12	13	B	C	16	1	14	15	2
A	D	9	10	11	12	13	14	15	16	0	1	2	B	C	5	6	7	3	4	8
D	15	16	0	1	2	3	4	5	6	7	8	B	C	11	12	A	13	9	10	14
4	5	6	7	8	9	10	11	12	13	14	B	C	0	1	A	D	2	15	16	3
11	12	13	14	15	16	0	1	2	3	B	C	6	7	A	D	10	8	4	5	9
1	2	3	4	5	6	7	8	9	B	C	12	13	A	D	16	0	14	10	11	15
8	9	10	11	12	13	14	15	B	C	1	2	A	D	5	6	7	3	16	0	4
15	16	0	1	2	3	4	B	C	7	8	A	D	11	12	13	14	9	5	6	10
5	6	7	8	9	10	B	C	13	14	A	D	0	1	2	3	4	15	11	12	16
12	13	14	15	16	B	C	2	3	A	D	6	7	8	9	10	11	4	0	1	5
2	3	4	5	B	C	8	9	A	D	12	13	14	15	16	0	1	10	6	7	11
9	10	11	B	C	14	15	A	D	1	2	3	4	5	6	7	8	16	12	13	0
16	0	B	C	3	4	A	D	7	8	9	10	11	12	13	14	15	5	1	2	6
6	B	C	9	10	A	D	13	14	15	16	0	1	2	3	4	5	11	7	8	12
B	C	15	16	A	D	2	3	4	5	6	7	8	9	10	11	12	0	13	14	1
C	4	5	A	D	8	9	10	11	12	13	14	15	16	0	1	B	6	2	3	7
10	11	A	D	14	15	16	0	1	2	3	4	5	6	7	B	C	12	8	9	13
7	1	12	6	0	11	5	16	10	4	15	9	3	14	8	2	13	A	B	C	D
13	7	1	12	6	0	11	5	16	10	4	15	9	3	14	8	2	C	D	A	B
3	14	8	2	13	7	1	12	6	0	11	5	16	10	4	15	9	D	C	B	A
14	8	2	13	7	1	12	6	0	11	5	16	10	4	15	9	3	B	A	D	C

It can be checked, moreover, that satisfying this condition ensures also that all the remaining values for the unknowns will be distinct. Making $x - 1 = u$, $y - 1 = v$ we get

$$v^4(u-1)(u^2+1) + v^3u(3u^2-3u+4) - v^2u^2(u^2-3u+6) - vu^3(u-4) - u^4 = 0.$$

For $u = 1$ the equation becomes

$$4v^3 - 4v^2 + 3v - 1 = 0,$$

which can be factorized

$$(2v-1)(2v^2-v+1) = 0.$$

However $u = 1$, $v = \frac{1}{2}$ gives $t_2 = t_4$, so we have to look for the roots of $2v^2 - v + 1 = 0$.

To solve that equation it is necessary that -7 be a quadratic residue, and this is so if $p \equiv 1, 2, 4 \pmod 7$.

Calling $i^2 = -7$, $u = 1$ gives $x = 2$, $y = (5 \pm i)/4$ and using $s_2 = 1$, $t_2 = 0$ we obtain as solution of the system

$$s_1 = \frac{1 \mp i}{4}, \qquad t_1 = \frac{1 \mp i}{2},$$

$$s_3 = \frac{3 \mp i}{2}, \qquad t_3 = 2,$$

$$s_4 = \frac{7 \mp 3i}{8}, \qquad t_4 = \frac{9 \mp 5i}{8}.$$

It is easy to check that 28 differences are not equal to zero for both values of i except for $p = 11$. In this case $s_2 = t_4 = 1$ for $i = -2$. However, using $i = 2$ we obtain $s_1 = 8$, $s_3 = 6$, $s_4 = 7$, $t_1 = 5$, $t_3 = 2$, $t_4 = 4$. Notice that Theorems 1 and 2 do not preclude the possibility of constructing three orthogonal squares using the method of sum composition since to each value of x correspond two values of y, except for $p = 11$. However, our attempts to construct three mutually orthogonal Latin squares using the method of sum composition failed thus far.

THEOREM 3. *If $n_2 \neq 6$ is even, then for any prime number $p \geqslant 2n_2$ it is always possible to compose $O(p^\alpha, 2)$ based on* GF(p^α) *with $O(n_2, 2)$ to obtain a $O(p^\alpha + n_2, 2)$ set.*

Proof. Consider the pattern

$$s_1 = K_h(s_2, t_2), \qquad t_1 = K_v(s_2, t_2),$$
$$s_2 = K_h(s_1, t_1), \qquad t_2 = K_v(s_1, t_1).$$

This system is solvable and will yield distinct solutions provided that the rank is two and

$$y\mu = 1 + \mu.$$

Taking $t_1 = s_1 + 1$ we obtain

$$s_2 = s_1 - \mu, \qquad t_2 = s_1 - y\mu = s_2 - 1;$$

that is, t_2, s_2 are also consecutive numbers. By properly choosing y, which uniquely determines x, since the equation of compatability is of first degree in s, we may achieve that $t_2 = t_1 + 1$; the choice is $\mu = -3$, which provides $y = \frac{2}{3}$ and $x = \frac{1}{2}$. The sets S and T are therefore

$$S = \{s_1, s_1 + 3\},$$
$$T = \{s_1 + 1, s_1 + 2\}.$$

By starting with $s_1 = 0$ and repeating the above process $n_2/2$ times, we obtain the sets of transversals

$$S = \{0, 3; 4, 7; \cdots; 2n_2 - 4, 2n_2 - 1\},$$
$$T = \{1, 2; 5, 6; \cdots; 2n_2 - 3, 2n_2 - 2\}.$$

We could also have considered the pattern

$$s_1 = K_h(s_2, t_2), \qquad t_1 = K_v(s_1, t_2),$$
$$s_2 = K_h(s_1, t_1), \qquad t_2 = K_v(s_2, t_1).$$

Taking s_1, t_1 as independent unknowns, the compatibility condition reduces to

$$y\mu(1 + \mu) = 1.$$

Using again $t_1 = s_1 + 1$ we obtain

$$s_2 = s_1 - \mu, \qquad t_2 = s_1 - (1 + \mu) = s_2 - 1;$$

that is, t_2, s_2 are also consecutive numbers; $t_2 = t_1 + 1$ would imply as before $\mu = -3$, $y = \frac{1}{6}$, $x = \frac{1}{4}$ and we will get

$$S = \{s_1, s_1 + 3\},$$
$$T = \{s_1 + 1, s_1 + 2\}.$$

Again by starting with $s_1 = 0$ and repeating the process $n_2/2$ times we obtain

$$S = \{0, 3; 4, 7; \cdots; 2n_2 - 4, 2n_2 - 1\},$$

$$T = \{1, 2; 5, 6; \cdots; 2n_2 - 3, 2n_2 - 2\};$$

however this time we have to reverse the order of the set T before projecting vertically.

Note that $xy = 1$, the condition used by Hedayat and Seiden for constructing orthogonal Latin squares using the method of sum composition, does not hold in this theorem. However, as in their work this theorem precludes obtaining more than two orthogonal Latin squares.

We shall conclude this paper showing that in some of the work of Hedayat and Seiden the condition $xy = 1$ was in fact necessary.

PROPOSITION. *If a pattern for composition of a $O(p^\alpha, 2)$ and a $O(3, 2)$ set is such that horizontal projection recovers transversals from both sets S and T, then $xy = 1$.*

Proof. Any of the six equations which determine the pattern, three will involve the function K_h and the other three equations will involve the function K_v. Adding the six equations we will always obtain, no matter what the pattern is,

$$\sum s_i + \sum t_i = (1 + \mu + 1 + y\mu) \sum s_i - (\mu + y\mu) \sum t_i$$

or

$$\left(\sum s_i - \sum t_i\right)(1 + \mu + y\mu) = 0.$$

If the horizontal projection recovers transversals from both S and T, adding the three equations involving K_h we will obtain in the l.h.s. the sum of either two s's and one t, or one s and two t's; in the r.h.s. we will obtain $\sum s_i - \mu(\sum t_i - \sum s_i)$. Therefore, if $\sum t_i - \sum s_i = 0$ we will have $s_i = t_j$ for some i, j. We must then have $1 + \mu + y\mu = 0$; but $1 + \mu + y\mu = xy - 1$, thus the result.

This proposition applies to 36 of the 48 possible patterns to compose $O(p^\alpha, 2)$ and $O(3, 2)$ sets; they have been fully investigated by Hedayat and Seiden.

Remark. The condition $(\sum s_i - \sum t_i)(1 + \mu + y\mu) = 0$ must hold for all patterns and is independent of the size of the system of equations involved. Hence, if we search for orthogonal Latin squares by the method of sum composition we must have either $xy = 1$ as assumed by Hedayat and Seiden or $\sum s_i = \sum t_i$, which will reduce the rank by at least 2.

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