Discrete Applied Mathematics 158 (2010) 1869–1878

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Exact values of $ex(\nu; \{C_3, C_4, \ldots, C_n\})$

E. Abajo, A. Diánez *

Departamento de Matemática Aplicada I, Universidad de Sevilla, Avda. Reina Mercedes 2, E-41012 Sevilla, Spain

A R T I C L E   I N F O

Article history:
Received 7 November 2007
Received in revised form 22 June 2010
Accepted 3 August 2010
Available online 1 September 2010

Keywords:
Extremal graphs
Girth
Cages

A B S T R A C T

For integers $n \geq 4$ and $\nu \geq n+1$, let $ex(\nu; \{C_3, C_4, \ldots, C_n\})$ denote the maximum number of edges in a graph with ν vertices and girth at least $n+1$. In this paper we have obtained bounds on this function for $n \in \{5, 6, 7\}$ and, in several cases, even the exact value. We have also developed a greedy algorithm for generating graphs with large size for given order and girth.

1. Introduction

For undefined terminology and notation we refer the reader to [8]. Let $V(G)$ and $E(G)$ denote respectively the set of vertices and the set of edges of a graph G. The size and order of G are denoted by $\nu(G)$ and $e(G)$ respectively and the number of vertices of degree i in G is denoted by n_i. For a vertex $x \in V(G)$ and an integer $i \geq 0$, the *neighborhood of x at distance i* is denoted by $N_i(x) = \{w \in G;\ d_G(x, w) = i\}$, where $d_G(x, w)$ is the length of the shortest (x, w)-path in G. Analogously, let $d_G(x, H) = \min\{d_G(x, u);\ u \in V(H)\}$ be the distance between a given vertex $x \in V(G)$ and a subgraph H in G. The longest distance between any two vertices in G is the diameter $D(G) = D$ of G. The *degree* of a vertex x is $\delta(x) = |N_1(x)|$ and the *minimum* and *maximum degree* over all vertices of G are denoted by $\delta(G) = \delta$ and $\Delta(G) = \Delta$, respectively.

The cycle of length n, $n \geq 3$, is referred to as C_n and the length of the shortest cycle in G as the *girth* of G or $g(G)$. Any graph without cycles is said to have an infinite girth. If $g(G) = g$ is odd, it is clear that the sets $N_i(x)$, $1 \leq i \leq (g-3)/2$, are independent and two vertices in $N_i(x)$ cannot have a common neighbor in $N_{i+1}(x)$. To study a graph G with even girth g, it is common to focus on an edge xy and consider the sets of vertices $X_i = \{u \in V(G);\ d_G(u, x) = i,\ d_G(u, y) = i+1\}$ and $Y_i = \{u \in V(G);\ d_G(u, y) = i,\ d_G(u, x) = i+1\}$, for $1 \leq i \leq (g-2)/2$. Notice that these $g-2$ sets are pairwise disjoint and each of them is independent. Also, two vertices of X_i or Y_i cannot have a common neighbor in X_{i+1} or Y_{i+1} for $1 \leq i \leq (g-4)/2$.

This paper deals with $ex(\nu; \{C_3, C_4, \ldots, C_n\})$, which represents the maximum number of edges in a simple graph of order ν and girth at least $n+1$. We refer to it as the extremal function. By $EX(\nu; \{C_3, C_4, \ldots, C_n\})$ we denote the set of all simple graphs of order ν, girth at least $n+1$ and with $ex(\nu; \{C_3, C_4, \ldots, C_n\})$ edges. Elements in $EX(\nu; \{C_3, C_4, \ldots, C_n\})$ are called extremal graphs.

It is well known (see [14]) that $ex(\nu; \{C_3\}) = \lfloor \nu^2/4 \rfloor$ and, therefore, we assume throughout this paper that $n \geq 4$. The values of $ex(\nu; \{C_3, C_4\})$ for all $\nu \leq 24$ are given in [10] and proofs of some of them appear in [11]. The corresponding ones for $25 \leq \nu \leq 30$ are determined in [12]. In the papers [10,17] the authors have implemented algorithms for constructing $\{C_3, C_4\}$-free graphs with as many edges as possible. These graphs provide lower bounds on $ex(\nu; \{C_3, C_4\})$ for orders $31 \leq \nu \leq 200$.

* Corresponding author. Fax: +34 954556621.
E-mail addresses: eabajo@us.es (E. Abajo), anadianez@us.es (A. Diánez).

0166-218X/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.dam.2010.08.002

Analogously, in [16], a more general algorithm is developed. Using this iterative process the authors provide lower bounds on the function $ex(\nu; \{C_3, C_4, \ldots, C_n\})$ for $n \in \{5, 6, 7\}$ and $\nu \leq 39$. We confirm in this paper that most of these lower bounds are the exact values of the corresponding extremal function. Recently in [1], using an idea inspired by the *excision method* used by Biggs in [6], the authors have improved some of the bounds on $ex(\nu; \{C_3, C_4\})$ included in [10] and also have found lower bounds on $ex(\nu; \{C_3, C_4, C_5, C_6\})$ for every order $40 \leq \nu \leq 300$.

The aim of this paper is to determine better and new bounds on the extremal function $ex(\nu; \{C_3, C_4, \ldots, C_n\})$ for $n \in \{5, 6, 7\}$ and, in some cases, we will show that they are the exact values of the extremal function.

2. Known results

For every $n \geq 4$ the extremal function $ex(\nu; \{C_3, C_4, \ldots, C_n\})$ is already determined when the order ν is not large.

Theorem 1. [3] *Let $n \geq 4$ and $0 \leq k \leq 4$ be integers. Then*

$$ex(\nu; \{C_3, C_4, \ldots, C_n\}) = \nu + k \quad \text{for each } \nu \in [\nu_k(n), \nu_{k+1}(n)),$$

where

$$\nu_0(n) = n + 1;$$
$$\nu_1(n) = \lfloor 3n/2 \rfloor + 1;$$
$$\nu_2(n) = 2n;$$
$$\nu_3(n) = \begin{cases} \lceil 9n/4 \rceil & \text{if } n \text{ is even} \\ \lfloor 9n/4 \rfloor & \text{if } n \text{ is odd;} \end{cases}$$
$$\nu_4(n) = \begin{cases} \lceil (8n-2)/3 \rceil & \text{if } n \text{ is even} \\ \lfloor (8n-2)/3 \rfloor & \text{if } n \text{ is odd;} \end{cases}$$
$$\nu_5(n) = \begin{cases} 3n-2 & \text{if } n \neq 6 \\ 17 & \text{if } n = 6. \end{cases}$$

The following result, demonstrated by Alon, Hoory and Linial, will allow us to obtain upper bounds on the extremal function $ex(\nu; \{C_3, C_4, \ldots, C_n\})$.

Theorem 2 ([4]). *For $g \geq 3$ and $d > 2$, put*

$$\nu_0(d, g) = \begin{cases} 1 + d \displaystyle\sum_{i=0}^{\frac{g-3}{2}} (d-1)^i & \text{if } g \text{ is odd;} \\ 2 \displaystyle\sum_{i=0}^{\frac{g-2}{2}} (d-1)^i & \text{if } g \text{ is even.} \end{cases}$$

A graph G, with average degree $\bar{d}$ and girth g, has at least $\nu_0(\bar{d}, g)$ vertices.

This result is well known for regular graphs (see [8], page 308). Among these kinds of graphs, we should remember that an (r, g)-*cage* is an r-regular graph with girth g and minimum order. In particular, an (r, g)-cage is said to be *minimal* if its order is exactly the previous lower bound $\nu_0(r, g)$. Cages and certain graphs in the family $EX(\nu; \{C_3, C_4, \ldots, C_n\})$ are related. A connection is provided next.

Theorem 3 ([2]). *Let $r \geq 3$ and $g \geq 5$ be given integers. If there is an (r, g)-cage with order $\nu_0(r, g)$, then*

$$EX(\nu_0(r, g); \{C_3, \ldots, C_{g-1}\}) = \{\text{minimal } (r, g)\text{-cages}\}.$$

The following three inequalities are useful in proving the new results that we have obtained.

Since the maximum degree is greater than the average degree, we have

$$\Delta(G) \geq \lceil 2e(G)/\nu(G) \rceil. \tag{1}$$

By removing one vertex of minimum degree from a given graph G with girth at least $n + 1$, we obtain

$$\delta(G) \geq e(G) - ex(\nu(G) - 1; \{C_3, C_4, \ldots, C_n\}). \tag{2}$$

It is well known (see [9]) that every graph G with $g(G) > n$ verifies

$$\nu(G) \geq \begin{cases} 1 + \sum_{i=1}^{\frac{n}{2}} \Delta(\delta-1)^{i-1} & \text{if } n \text{ is even} \\ 1 + \sum_{i=1}^{\frac{n-1}{2}} \Delta(\delta-1)^{i-1} + (\delta-1)^{\frac{n-1}{2}} & \text{if } n \text{ is odd.} \end{cases} \tag{3}$$

3. A summary of our main results

Now we present the main results of this paper. Although we include tables of exact values and bounds on $ex(\nu; \{C_3, C_4, \ldots, C_n\})$ for $n \in \{5, 6, 7\}$, we would like to remark that this extremal function is already known (see [2,3]) for the following cases:

$n = 5, \quad \nu \in \{1, 2, \ldots, 12, 14, 26, 42, 62\};$

$n = 6, \quad \nu \leq 16;$

$n = 7, \quad \nu \in \{1, 2, \ldots, 18, 30, 80\}.$

Theorem 4. *The extremal function $ex(\nu; \{C_3, C_4, C_5\})$ has the following values:*

ν	0	1	2	3	4	5	6	7	8	9
0	0	0	1	2	3	4	6	7	9	10
10	12	14	16	18	21	22	24	26	29	31
20	34	36	39	42	45	48	52	53	56	58
30	61	64	67	70	74	77	81	84	88	92
40	96	100	105	106–108	108–112	110–116	114–119	118–123	122–127	125–131
50	130–135	134–139	138–143	142–147	147–151	151–155	156–160	160–164	165–168	170–172
60	175–177	180–181	186							

Theorem 5. *The extremal function $ex(\nu; \{C_3, C_4, C_5, C_6\})$ has the following values:*

ν	0	1	2	3	4	5	6	7	8	9
0	0	0	1	2	3	4	5	7	8	9
10	11	12	14	15	17	18	20	22	23	25
20	27	29	31	33	36	37	39	41	43	44–47
30	47–49	48–52	50–54	52–56	55–58	57–61	59–63	61–65	62–68	64–70
40	67–73	69–75	71–77	73–80	75–82	77–85	80–87	82–90	84–92	87–95

Corollary 6.

$EX(24; \{C_3, C_4, C_5, C_6\}) = \{(3, 7)\text{-}McGee\ graph\}$

Theorem 7. *The extremal function $ex(\nu; \{C_3, C_4, \ldots, C_7\})$ has the following values:*

ν	0	1	2	3	4	5	6	7	8	9
0	0	0	1	2	3	4	5	6	8	9
10	10	12	13	14	16	18	19	20	22	24
20	25	27	29	30	32	34	36	38	40	42
30	45	46	47	49	51	53	55	56–59	58–61	60–63
40	62–65	64–67	65–69	67–71	69–73	71–76	73–78	75–80	77–82	79–84
50	81–87	84–89	86–91	88–93	90–96	93–98	96–100	98–103	100–105	102–107
60	105–110	108–112	110–115	112–117	114–119	117–122	120–124	122–127	125–129	128–132
70	130–134	133–137	136–139	138–142	141–144	144–147	147–149	150–152	153–154	156–157
80	160									

Theorem 3 [2] connects two different extremal problems in graphs, establishing that the elements of the family $EX(\nu_0(r, g); \{C_3, C_4, \dots, C_{g-1}\})$ are exactly the minimal (r, g)-cages whenever these graphs exist. This result, together with the uniqueness of the (3, 6)-, (4, 6)-, (5, 6)-, (6, 6)-, (3, 8)-, (4, 8)-cages, guarantees that these graphs are the single elements of the corresponding extremal sets. Similarly, Corollary 6 asserts that the (3, 7)-McGee graph is the unique element in $EX(24; \{C_3, C_4, C_5, C_6\})$ although it is not a minimal cage.

4. Searching for lower and upper bounds

In our proof, to show that $ex(\nu; \{C_3, C_4, \dots, C_n\}) = m$ we need the construction of at least one graph of order ν, size m and girth at least $n + 1$. We have developed an iterative greedy process that constructs graphs with large size and given girth:

Algorithm 1. LSG(G_0, ν^*)

Input: A $\{C_3, C_4, \dots, C_n\}$-free graph G_0 with diameter $D(G_0) = D_0 < n$ and a positive integer $\nu^* > \nu(G_0)$.

Output: A graph G^* such that $\nu(G^*) = \nu^*,\ g(G^*) > n,\ D(G^*) < n$ and $e(G^*) \geq e(G_0) + \nu^* - \nu(G_0)$.

Step 0: Make $G = G_0$ and $D = D_0$.

Step 1: If $\nu^* - \nu(G) < n - D$, then G^* is the graph obtained by subdividing, $\nu^* - \nu(G)$ times, an arbitrary edge of G. Otherwise, we consider all the graphs obtained by adding to G a path of length $n - D + 1$ between two vertices in $V(G)$ at distance D. We choose G' as one of them with the longest diameter.

Step 2: While $D(G') \geq n$, we insert in G' an edge between two vertices at maximum distance in G'. Again, these two vertices must be always chosen such that the resulting graph, also denoted by G', has the longest diameter. We make $G = G',\ D = D(G')$ and go to Step 1.

Notice that the input of the previous algorithm can be an extremal $\{C_3, C_4, \dots, C_n\}$-free graph, since it is proved in [5] that all of these graphs have diameter at most $n - 1$. Most of the extremal graphs and the ones which achieve the lower bounds in Theorems 4, 5 and 7 can be obtained by using Algorithm 1. We would like to remark that this iterative process sometimes provides better results than the one developed in [16].

In our attempt to construct graphs with maximum size for given order and girth, we see that extremal graphs of orders ν and $\nu - 1$ are sometimes related.

Remark 8. If G is any $\{C_3, C_4, \dots, C_n\}$-free graph with order ν, the graph obtained by removing one vertex in G is a $\{C_3, C_4, \dots, C_n\}$-free graph with order $\nu - 1$.

Applying Remark 8 we have succeeded in constructing some extremal graphs, since it appears that the removal of vertices in extremal graphs produces new extremal graphs in most of the cases. In particular, we have applied this idea to the (3, 6)-, (3, 8)-, (4, 6)-, (4, 8)-, (5, 6)-, (6, 6)-cages to find most of the extremal graphs and the graphs which provide the lower bounds in the constructive proofs of Theorems 4, 5 and 7.

The given upper bounds in Theorems 4, 5 and 7 are obtained from Theorem 2 [4]. If there exists a graph G with given order and girth, but its size is larger than the one established in Theorems 4, 5 and 7, then G will have an average degree greater than the one stated in Theorem 2 [4].

5. Proofs

In analogy to paper [10], for convenience, we define $ex(\nu; \{C_3, C_4, \dots, C_n\})$ by $f_n(\nu)$ and the family of the $\{C_3, C_4, \dots, C_n\}$-free graphs with order ν by $\mathcal{F}_\nu^n$. Given $n \geq 4$ and $\nu \geq n + 1$, to show that $f_n(\nu) = m$, we first prove that $f_n(\nu) \geq m$ and generate, with the help of Algorithm 1 and Remark 8, a graph $G \in \mathcal{F}_\nu^n$ with m edges. Next, we assume that there exists a graph $G \in \mathcal{F}_\nu^n$ with size $m + 1$. Making use of the inequalities (1)–(3), we obtain the possible values of $\delta(G)$ and $\Delta(G)$, and considering the known previous values of $f_n(\nu)$ we reach a contradiction. To prove that $f_n(\nu) \leq m$, we must notice that it is not necessary to consider the existence of a graph G with $m + k$ edges for $k > 1$, because by removing edges from G, it would also be possible to construct a graph with the same order, exactly $m + 1$ edges and at least the same girth.

We consider a special edge in a graph free from triangles and quadrilaterals.

Definition 9. Let G be a $\{C_3, C_4\}$-free graph and x a vertex in $V(G)$ such that $\delta(x) = \Delta$ and $|N_2(x)|$ is as large as possible. An edge xy in $E(G)$ is said to be a distinguished edge of G if y is a vertex of maximum degree in $N_1(x)$ such that $\sum_{u \in N_1(y)} \delta(u)$ is maximum.

We illustrate this special edge in Fig. 1.

Proof of Theorem 4. Notice that Theorem 1 [3] provides the value of $f_5(\nu) = ex(\nu; \{C_3, C_4, C_5\})$ for orders $\nu \leq 12$ and that Theorem 3 [2] establishes $f_5(14) = 21,\ f_5(26) = 52,\ f_5(42) = 105$ and $f_5(62) = 186$.

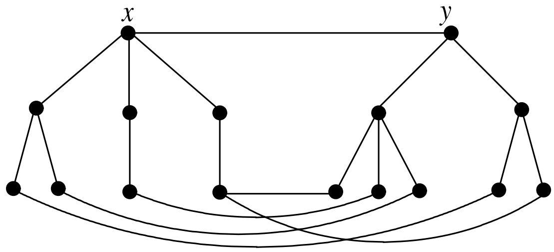

Fig. 1. Illustration of a distinguished edge of a graph.

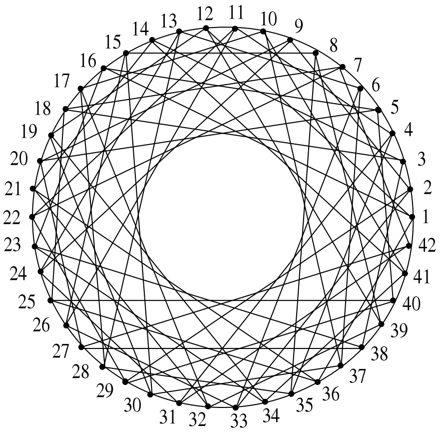

Fig. 2. The removal of vertices from the (5, 6)-cage provides graphs in $EX(\nu; \{C_3, C_4, C_5\})$ for $28 \le \nu \le 41$.

Table 1
Adjacency list of the (6, 6)-cage.

$1 \to \{26, 31, 36, 41, 46, 51\}$	$2 \to \{27, 35, 37, 42, 47, 51\}$	$3 \to \{28, 34, 38, 43, 48, 51\}$
$4 \to \{29, 33, 39, 44, 49, 51\}$	$5 \to \{30, 32, 40, 45, 50, 51\}$	$6 \to \{26, 32, 38, 44, 47, 52\}$
$7 \to \{27, 31, 39, 45, 48, 52\}$	$8 \to \{28, 35, 40, 41, 49, 52\}$	$9 \to \{29, 34, 36, 42, 50, 52\}$
$10 \to \{30, 33, 37, 43, 46, 52\}$	$11 \to \{26, 33, 40, 42, 48, 53\}$	$12 \to \{27, 32, 36, 43, 49, 53\}$
$13 \to \{28, 31, 37, 44, 50, 53\}$	$14 \to \{29, 35, 38, 45, 46, 53\}$	$15 \to \{30, 34, 39, 41, 47, 53\}$
$16 \to \{26, 34, 37, 45, 49, 54\}$	$17 \to \{27, 33, 38, 41, 50, 54\}$	$18 \to \{28, 32, 39, 42, 46, 54\}$
$19 \to \{29, 31, 40, 43, 47, 54\}$	$20 \to \{30, 35, 36, 44, 48, 54\}$	$21 \to \{26, 35, 39, 43, 50, 55\}$
$22 \to \{27, 34, 40, 44, 46, 55\}$	$23 \to \{28, 33, 36, 45, 47, 55\}$	$24 \to \{29, 32, 37, 41, 48, 55\}$
$25 \to \{30, 31, 38, 42, 49, 55\}$	$26 \to \{1, 6, 11, 16, 21, 56\}$	$27 \to \{2, 7, 12, 17, 22, 56\}$
$28 \to \{3, 8, 13, 18, 23, 56\}$	$29 \to \{4, 9, 14, 19, 24, 56\}$	$30 \to \{5, 10, 15, 20, 25, 56\}$
$31 \to \{1, 7, 13, 19, 25, 57\}$	$32 \to \{5, 6, 12, 18, 24, 57\}$	$33 \to \{4, 10, 11, 17, 23, 57\}$
$34 \to \{3, 9, 15, 16, 22, 57\}$	$35 \to \{2, 8, 14, 20, 21, 57\}$	$36 \to \{1, 9, 12, 20, 23, 58\}$
$37 \to \{2, 10, 13, 16, 24, 58\}$	$38 \to \{3, 6, 14, 17, 25, 58\}$	$39 \to \{4, 7, 15, 18, 21, 58\}$
$40 \to \{5, 8, 11, 19, 22, 58\}$	$41 \to \{1, 8, 15, 17, 24, 59\}$	$42 \to \{2, 9, 11, 18, 25, 59\}$
$43 \to \{3, 10, 12, 19, 21, 59\}$	$44 \to \{4, 6, 13, 20, 22, 59\}$	$45 \to \{5, 7, 14, 16, 23, 59\}$
$46 \to \{1, 10, 14, 18, 22, 60\}$	$47 \to \{2, 6, 15, 19, 23, 60\}$	$48 \to \{3, 7, 11, 20, 24, 60\}$
$49 \to \{4, 8, 12, 16, 25, 60\}$	$50 \to \{5, 9, 13, 17, 21, 60\}$	$51 \to \{1, 2, 3, 4, 5, 61\}$
$52 \to \{6, 7, 8, 9, 10, 61\}$	$53 \to \{11, 12, 13, 14, 15, 61\}$	$54 \to \{16, 17, 18, 19, 20, 61\}$
$55 \to \{21, 22, 23, 24, 25, 61\}$	$56 \to \{26, 27, 28, 29, 30, 62\}$	$57 \to \{31, 32, 33, 34, 35, 62\}$
$58 \to \{36, 37, 38, 39, 40, 62\}$	$59 \to \{41, 42, 43, 44, 45, 62\}$	$60 \to \{46, 47, 48, 49, 50, 62\}$
$61 \to \{51, 52, 53, 54, 55, 62\}$	$62 \to \{56, 57, 58, 59, 60, 61\}$	

As we have pointed out, this proof is constructive and for each order ν the construction of at least one $\{C_3, C_4, C_5\}$-free graph with ν vertices is required. The size of this graph provides a lower bound on the extremal function $ex(\nu; \{C_3, C_4, C_5\})$. For orders $\nu \le 27$ we consider the graph included in [16]. For $28 \le \nu \le 41$ we construct the desired graphs by removing from the (5, 6)-cage (see Fig. 2) one by one and in the specified order the vertices of the set {29, 30, 31, 32, 39, 40, 2, 33, 37, 6, 5, 4, 3, 38}. Analogously, the removal of the vertices {61, 51, 1, 26, 56, 62, 27, 2, 60, 47, 46, 22, 44, 6, 55, 21, 59} from the (6, 6)-cage (see Table 1) provides the given lower bounds for $45 \le \nu \le 61$. For $\nu \in \{43, 44\}$, the graphs are obtained by applying Algorithm 1 to the (5, 6)-cage.

Next we prove that, for every order $\nu \in \{13, 15, 16, \ldots, 25, 27, 28, \ldots, 41\}$, the corresponding graph belongs to $EX(\nu; \{C_3, C_4, C_5\})$ and hence its size provides the exact value of $f_5(\nu)$.

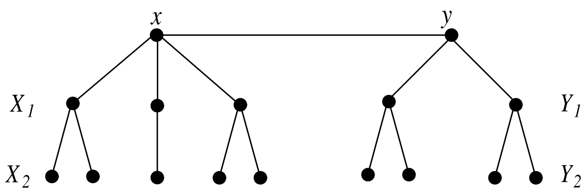

Fig. 3. Illustration of the non-existence in $\mathcal{F}_{15}^5$ of a graph with 23 edges.

$f_5(13) = 18$. If we assume that there exists a graph with 13 vertices, 19 edges and girth at least 6, it will contradict the inequality (3) for the values of δ and Δ provided by (1) and (2).

The proof is similar if $\nu \in \{16, 18, 20, 22, 23, 24, 25, 28, 34, 36, 38, 39, 40, 41\}$.

$f_5(15) = 22$. Every graph G in the set $\mathcal{F}_{15}^5$ with 23 edges has average degree strictly greater than 3 and hence $\Delta(G) \geq 4$. Since $f_5(13) = 18$, the removal of two vertices from G must imply the removal of at least five edges. Therefore, G contains at most one vertex of degree 2, which cannot be adjacent to a vertex of degree 3. Considering a distinguished edge xy, it is straightforward to see that $\nu(G) \geq 2 + |X_1| + |X_2| + |Y_1| + |Y_2| \geq 16$, a contradiction to the fact that the order of G is 15, as is shown in Fig. 3.

$f_5(17) = 26$. We assume that there exists a graph $G \in \mathcal{F}_{17}^5$ with 27 edges. In this case, $\delta = 3$, $\Delta = 4$ and $n_4 = 3$. We consider an arbitrary distinguished edge xy. Since $\nu(G) = 17$, the vertex y and every vertex in $X_1 \cup Y_1$ must have degree 3. Then, $e(G) = 16 + \sum_{u \in Y_2} (\delta(u) - 1) \leq 26$.

The same line of reasoning can be applied when $\nu = 30$.

$f_5(19) = 31$. Every graph $G \in \mathcal{F}_{19}^5$ with size 32 satisfies the conditions $\delta = 3$, $\Delta = 4$ and $n_3 = 12$. Two vertices of degree 3 in $V(G)$ cannot be adjacent, because otherwise, removing these two vertices would produce a graph which contradicts the assertion that $f_5(17) = 26$. Hence, the three edges incident in each of the twelve vertices of degree 3 in G are different, implying that $e(G) \geq 36$.

$f_5(21) = 36$. We assume that there exists a graph $G \in \mathcal{F}_{21}^5$ with 37 edges. Then, $\delta = 3$ and $4 \leq \Delta \leq 5$. Since $f_5(19) = 31$, two vertices of degree 3 in $V(G)$ cannot be adjacent. By xy we denote a distinguished edge in $E(G)$. If $\Delta = 5$, then $\nu(G) \geq 2 + |X_1| + |X_2| + |Y_1| + |Y_2| \geq 22$ and the order of G is contradicted. Therefore, $\Delta = 4$ and G has eleven vertices of degree 4. Each of them cannot be adjacent to three vertices of degree 3, because otherwise the removal of these four vertices contradicts the assertion that $f_5(17) = 26$. In this case, for the distinguished edge xy, we have $\nu(G) \geq 2 + |X_1| + |X_2| + |Y_1| + |Y_2| \geq 22$.

$f_5(27) = 53$. Every graph $G \in \mathcal{F}_{27}^5$ with 54 verifies that $2 \leq \delta \leq 4$. If $\delta = 4$, then G is regular. In this case, for an arbitrary edge $xy \in E(G)$, there exists one vertex s which is located at distance 3 from it. Since G contains no quadrilateral, the vertex s must be adjacent both to X_2 and to Y_2. Then, it can be assumed that $E(G)$ contains edges su_1, su_2, sv such that u_1, u_2 belongs to X_2 and v to Y_2. Since $\delta(v) = 4$, the vertex v must be adjacent to a vertex $r \in X_2$, such that either $d_G(r, u_1) = 2$ or $d_G(r, u_2) = 2$. However, this implies that G contains a cycle of length 5.

If $\delta = 3$, then $5 \leq \Delta \leq 7$. From the assertions that $f_5(25) = 48$, $f_5(24) = 45$, $f_5(23) = 42$, $f_5(22) = 39$ and $f_5(21) = 36$, it follows that every vertex x in $V(G)$ is adjacent at most to $\delta(x) - 3$ vertices of degree 3. We consider a distinguished edge xy. With 27 vertices, there is a single configuration with girth at least 6 that has the previously mentioned property. This means that $\delta(x) = 5$, as x is adjacent to two vertices of degree 3 and to three vertices of degree 4 (y one of them), and the vertex y is adjacent to two vertices of degree 4, one vertex of degree 3 and one of degree 5 (the vertex x). Therefore, $\Delta = 5$ and every vertex of $V(G)$ with maximum degree is adjacent to two vertices of degree 3. From $e(G) = 54$, we obtain that $\sum_{u \in X_2} (\delta(u) - 1) = 28$ and, since X_2 contains no vertex of degree 5, there are exactly two vertices of degree 3 in X_2. From the assertion that $n_5 = n_3$, it follows that there are at least four vertices of degree 5 in Y_2. Each of them must be adjacent to the two vertices of degree 3 in X_2 and consequently G contains a quadrilateral. If $\delta = 2$, then $\Delta \geq 5$. Taking into account that $f_5(25) = 48$, the removal of two vertices in $V(G)$ implies the removal of at least six edges in $E(G)$. Therefore, in $V(G)$ there is only one vertex of degree 2 and no vertex of degree 3. In this case, for a distinguished edge xy we have $\nu(G) \geq 2 + |X_1| + |X_2| + |Y_1| + |Y_2| \geq 28$, a contradiction.

$f_5(29) = 58$. We assume that there is a graph $G \in \mathcal{F}_{29}^5$ with 59 edges. In this case, $\delta = 3$ and $5 \leq \Delta \leq 8$. We denote by xy a distinguished edge in $E(G)$ and deal with several cases depending on the maximum degree of G. If $\Delta \geq 6$, then $\delta(x) \geq 6$ and all the vertices adjacent to x cannot have degree 3, because that would contradict one of the above proved assertions that $f_5(22) = 39$, $f_5(21) = 36$, $f_5(20) = 34$. Consequently, $\delta(y) = 5$ or $\delta(y) = 4$. Since G has 29 vertices, $\delta(y) = 5$ implies that all the vertices in $X_1 \cup Y_1$ have degree 3, and $\delta(y) = 4$ that at least four vertices in $X_1 \cup Y_1$ have minimum degree. In both cases, the removal of the vertices x, y and that of the ones in $X_1 \cup Y_1$ contradict that $f_5(18) = 29$ and $f_5(22) = 39$, respectively. Therefore, $\Delta = 5$ and $n_5 = n_3 + 2$. As $\nu(G) = 29$, the vertices x, y cannot have degree 5, because that implies that there are at least five vertices of degree 3 located at distance 1 from the edge xy, contradicting that $f_5(22) = 39$. Analogously, if we assume that x is only adjacent to vertices of degree 4, each of them must be adjacent to at least one vertex of degree 3 and, from $f_5(21) = 36$, they cannot be adjacent to two vertices of degree 3. However this implies that $e(G) = 28 + \sum_{u \in X_2} (\delta(u) - 1) = 60$, because there is no vertex of maximum degree in X_2. Therefore, every vertex of degree 5 must be adjacent to at least $\alpha \geq 1$ vertices of degree 3. We distinguish several cases according to α.

When we consider that $\alpha \geq 3$, the equality $n_5 = n_3 + 2$ is incompatible with $3n_5 \leq |\{uv \in E(G);\ \delta(u) = 3,\ \delta(v) = 5\}| \leq |\{uv \in E(G);\ \delta(u) = 3\}| \leq 3n_3$. Let us study the case $\alpha = 2$. First, we assume that there is one vertex of degree 5 in Y_1 and consequently at least two vertices of degree 3 in Y_2. Since $f_5(21) = 36$, the set Y_1 contains no vertex of degree 3. Therefore, $2 + |X_1| + |Y_1| + |X_2| + |Y_2| = 29$ and $e(G) = 28 + \sum_{u \in X_2}(\delta(u) - 1) \leq 58$ due to X_2 containing no vertex of degree 5. Hence, if $\alpha = 2$ there exists no vertex of degree 5 in Y_1. As $f_5(23) = 42$, $f_5(22) = 39$ and X_1 has exactly two vertices of degree 3, the set $N_2(x)$ has at most one vertex of degree 3. Therefore, by definition of the distinguished edge, the vertices in Y_1 have degree 4. From $f_5(22) = 39$ and $f_5(21) = 36$, it follows that there is at most one vertex of degree 3 in $X_2 \cup Y_2$, and consequently there are at most two vertices of degree 3 which are located at distance at least 2 from the edge xy. From $n_3 \geq \alpha = 2$, we obtain that $n_5 = n_3 + 2 \geq 4$ and hence there are at least three vertices of degree 5 located at distance at least 2 from the edge xy. But these vertices cannot be adjacent to at least $\alpha = 2$ vertices of degree 3 without forming quadrilaterals.

Finally, we assume that $\alpha = 1$. As $f_5(21) = 36$ and $f_5(22) = 39$, and there are at most two vertices of degree 3 in $N_2(x)$, and consequently we can assume that Y_1 contains no vertex of degree 3. Taking into account that $\nu(G) = 29$, we obtain that Y_1 contains no vertex of degree 5 and that $2 + |X_1| + |Y_1| + |X_2| + |Y_2| = 29$. This implies that there is no vertex of degree 5 in X_2. Since $\sum_{u \in X_2}(\delta(u) - 1) = 31$, exactly two vertices of degree 3 belong to X_2. Notice that we have just seen that every vertex in $V(G)$ of degree 5 with exactly one vertex of degree 3 in its neighborhood has no vertex of degree 5 at distance 2. From $n_3 \geq 3$, it follows that $n_5 \geq 5$, and consequently the set Y_2 contains two vertices of degree 5 located at distance 2. Both of them must be adjacent to the two vertices of degree 3 in X_2 and, therefore, the graph G contains a quadrilateral.

$f_5(31) = 64$. Every graph $G \in \mathcal{F}^5_{31}$ with 65 edges has $\delta = 4$, $\Delta = 5$ and $n_5 = 6$. Let xy denote an arbitrary distinguished edge of G. From $\nu(G) = 31$, we obtain that $\delta(y) = 4$ and that every vertex of degree 4 is adjacent to at most two vertices of maximum degree. Therefore, Y_1 contains at most one vertex of degree 5. But its existence implies $e(G) = 30 + \sum_{u \in Y_2}(\delta(u) - 1) \leq 64$. Hence, two vertices of degree 5 must be at least at distance 3 apart. This implies that there is no vertex of degree 5 in $X_2 \cup Y_1$ and also that there are two vertices of degree 5 at distance 2 in Y_2, contradicting the previous assertion.

$f_5(32) = 67$. Let us assume that there is a graph $G \in \mathcal{F}^5_{32}$ with size 68. Clearly, $\delta = 4$, $\Delta = 5$ and $n_5 = 8$. Two vertices of degree 5 in $V(G)$ cannot be adjacent, because for a distinguished edge we would have $\nu(G) \geq 34$. We denote by α the maximum number of vertices of degree 5 adjacent to every vertex of degree 4. On considering xy a distinguished edge in $V(G)$, the case $\alpha = 4$ implies that $\nu(G) \geq 33$. Also, as a consequence of the non-existence of vertices with maximum degree in X_2, when $\alpha = 3$, we have $e(G) = 31 + \sum_{u \in X_2}(\delta(u) - 1) = 67$.

On the other hand, if $\alpha = 1$, all the vertices adjacent to vertices of degree 5 are different, and therefore, there must be at least forty vertices of degree 4 in $V(G)$. Hence $\alpha = 2$. By using a distinguished edge xy we confirm that X_1 has only vertices of degree 4, that there is a single vertex of maximum degree in Y_1 and also that there is a vertex s located at distance 3 from the edge xy. Since $\alpha = 2$, there are at most four vertices of degree 5 in the set Y_2. Considering that every vertex of maximum degree in X_2 must be adjacent to s, together with the equalities $\alpha = 2$ and $n_5 = 8$, we obtain that $\delta(s) = 4$ and that there are exactly two vertices with degree 5 in X_2. Both of them must be adjacent to s and to a vertex v in Y_2 with degree 4 and located at distance 2 from two vertices in Y_2 with degree 5. However this means that G contains a quadrilateral.

$f_5(33) = 70$. The proof is similar to the previous one. We assume that there is a graph $G \in \mathcal{F}^5_{33}$ with 71 edges. Clearly, $\delta = 4$, $\Delta = 5$ and $n_5 = 10$. Let xy denote a distinguished edge in $E(G)$. Two vertices of degree 5 cannot be adjacent, because otherwise the order of G should be at least 34. It makes sense to denote by α the maximum number of vertices of degree 5 adjacent to any vertex of degree 4. If $\alpha = 4$, then $2 + |X_1| + |X_2| + |Y_1| + |Y_2| = \nu(G)$ and taking into account that there is no vertex of degree 5 in X_2, we obtain that $e(G) = 32 + \sum_{u \in X_2}(\delta(u) - 1) = 68$.

On the other hand, if $\alpha = 1$, all the vertices of degree 4 adjacent to vertices of degree 5 are different, and hence there must be 50 vertices in G with minimum degree.

We consider that $\alpha = 3$. In this case, the set X_1 has only vertices of degree 4, there are two vertices of degree 5 in Y_1 and there exists one vertex s located at distance 3 from the edge xy. Taking into account that in Y_2 there are at most three vertices of degree 5 and that every vertex of maximum degree in X_2 is adjacent to s, we conclude that the vertex s has degree 4 and that there are at most $\alpha = 3$ vertices of degree 5 in X_2. This contradicts that $n_5 = 10$. Therefore, $\alpha = 2$. In this case, the set X_1 has only vertices of minimum degree, there is one vertex of degree 5 in Y_1 and there are two vertices s, t at distance 3 from the edge xy. Since in the set Y_2 there are at most four vertices of degree 5 and every vertex of maximum degree in X_2 is adjacent either to s or to t, it follows, from $\alpha = 2$, that $\delta(s) = \delta(t) = 4$ and that there are in X_2 four vertices of degree 5, two of them adjacent to s and the other ones to t. These two vertices are also adjacent to a vertex v in Y_2 with degree 4 and located at distance 2 from two vertices in Y_2 with degree 5. This implies that G contains a quadrilateral.

$f_5(35) = 77$. Every graph $G \in \mathcal{F}^5_{35}$ with size 78 satisfies the conditions $\delta = 4$ and $5 \leq \Delta \leq 6$. From $f_5(33) = 70$, it follows that two vertices of degree 4 in $V(G)$ cannot be adjacent. Moreover, Δ must be 5, because considering a distinguished edge, when $\Delta = 6$, we have $\nu(G) \geq 37$. Consequently, $V(G)$ contains 19 vertices of degree 4, each of them adjacent to vertices of degree 5. Therefore $|\{uv \in E(G);\ \delta(u) = 4, \delta(v) = 5\}| = 4n_4 = 76$ and there are in $E(G)$ only two edges joining vertices of degree 5. Let us denote one of them by xy. The set $X_1 \cup Y_1$ contains at least seven vertices of degree 4 and hence there are in $X_2 \cup Y_2$ at least 21 vertices of maximum degree, contradicting that $n_5 = 16$ in G.

$f_5(37) = 84$. Every graph $G \in \mathcal{F}^5_{37}$ with size 85 satisfies $\delta = 4$ and $5 \leq \Delta \leq 6$. From $f_5(35) = 77$ and $f_5(33) = 70$, it follows that two vertices of degree 4 in G cannot be adjacent and that every vertex of degree 5 is adjacent to at most two vertices

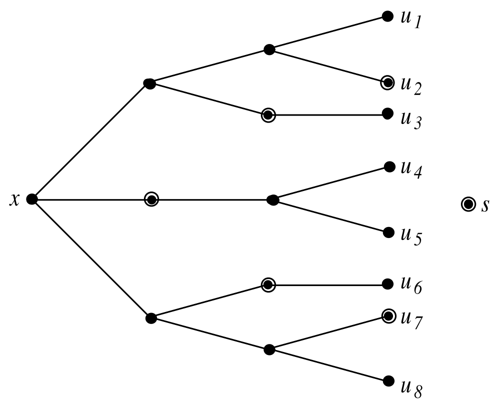

Fig. 4. The subgraph H mentioned in the proof of the equality $f_6(18) = 23$. Let us note that vertices of degree 2 are highlighted.

of degree 4. Let xy be a distinguished edge of G. First we assume that $\Delta = 6$. As $\nu(G) = 37$, every vertex in the set $\{y\} \cup X_1$ has degree 4 and the ones in $X_2 \cup Y_1$ have degree 5. That is the only possible configuration of G, but it contains at most nine vertices of degree 4, contradicting that $n_4 = n_6 + 15$. Hence, $\Delta = 5$ and considering a distinguished edge we obtain that $\nu(G) \geq 38$. $\square$

Theorem 4 confirms that the lower bounds on $ex(\nu; \{C_3, C_4, C_5\})$ provided in [16] are optimal for orders $\nu \leq 27$ and $\nu = 30, 32$.

Proof of Theorem 5. All the graphs that achieve the exact values and the lower bounds on $ex(\nu; \{C_3, C_4, C_5, C_6\})$ for $\nu \neq 30$ are constructed using Algorithm 1 beginning with the cycle C_7. We would like to acknowledge that for $\nu = 30$ this iterative process produces a graph with size 46, while in [16] another one is constructed with one more edge.

Next, we confirm the extremality of these graphs for $17 \leq \nu \leq 28$. Notice that Theorem 1 [3] provides the value of $f_6(\nu)$ for $\nu \leq 16$.

$f_6(17) = 22$. Every graph with 17 vertices, 23 edges and girth at least 7 contradicts the inequality (3) for the values of δ and Δ provided by (1) and (2).

The proof is similar when $\nu \in \{19, 20, 21, 24, 26, 27, 28\}$.

$f_6(18) = 23$. We assume that there exists a graph $G \in \mathcal{F}^6_{18}$ with size 24. Then, $\delta = 2$ and $3 \leq \Delta \leq 5$. Taking into account that $f_6(16) = 20$, $f_6(15) = 18$ and $f_6(13) = 15$, we verify that the graph G does not contain two consecutive vertices of minimum degree, that every vertex of degree 3 is adjacent to at most one vertex of degree 2 and that every vertex of degree 4 is adjacent to at most three vertices of degree 2. Then, if $\Delta \geq 4$, for every vertex x such that $\delta(x) = \Delta$, we have $\nu(G) \geq 1 + |N_1(x)| + |N_2(x)| + |N_3(x)| \geq 19$. Hence $\Delta = 3$, $n_2 = 6$, $n_3 = 12$ and every vertex of degree 3 in $V(G)$ is adjacent to exactly one vertex of degree 2. Therefore, for every vertex x of degree 3 it is clear that $\bigcup_{i=0}^{3} |N_i(x)| = 17$ and so there is a vertex s in $V(G)$ such that $d_G(x, s) = 4$. Hence, G contains the subgraph H included in Fig. 4, in which the vertices in $N_3(x)$ are denoted by $u_1, \ldots, u_8$. We assume that u_2, u_7 have degree 2. Clearly, $\delta(s) = 2$. The vertex u_6 has degree 3 and without loss of generality we consider that $u_6u_5 \in E(G)$. If u_6 is adjacent to u_3, then $\delta(u_3) = 3$ and $g(G) \geq 7$ are incompatible. Consequently, the edge $u_6u_1 \in E(G)$ and this implies that the edge u_1s also belongs to $E(G)$. In this case, the vertex u_5 cannot be adjacent to any vertex of degree 2 without avoiding the forbidden cycles C_4, C_5 and C_6.

$f_6(22) = 31$. It is not possible to construct a graph $G \in \mathcal{F}^6_{22}$ with 32 edges, since its minimum degree $\delta(G) \leq \bar{d}(G) = 64/22 < 3$ contradicts the inequality (2).

The same line of reasoning can be applied when $\nu = 23$.

$f_6(25) = 37$. Every graph in the set $\mathcal{F}^6_{25}$ with 38 edges satisfies the conditions $\delta \geq 2$ and $\Delta \geq 4$. From $f_6(23) = 33$, it follows that $n_2 \leq 1$. Considering the neighborhoods of a vertex x of maximum degree we obtain that $\nu(G) \geq 1 + |N_1(x)| + |N_2(x)| + |N_3(x)| \geq 26$. $\square$

Theorem 5 confirms that the lower bounds on $f_6(\nu)$ provided in the paper [16] are optimal for every $\nu \leq 28$ except for $\nu = 26$.

Proof of Corollary 6. In Theorem 5 we have obtained that $ex(24; \{C_3, C_4, C_5, C_6\}) = 36$. This implies that every graph G in $EX(24; \{C_3, C_4, C_5, C_6\})$ has average degree 3. Considering also the inequality (2) we have $\delta(G) \geq e(G) - ex(23; \{C_3, C_4, C_5, C_6\}) = 3$ and, therefore, the graph G is 3-regular. McGee constructed such $(3, 7)$-cage for the first time in [13] and Tutte proved its uniqueness in [15]. $\square$

Proof of Theorem 7. Notice that Theorem 1 [3] provides the value of $f_7(\nu)$ for orders $\nu \leq 18$ and that Theorem 3 [2] establishes $f_7(30) = 45$ and $f_7(80) = 160$. All the given exact values and the lower bounds on $ex(\nu; \{C_3, C_4, \ldots, C_7\})$ for $\nu \leq 46$ are obtained by Algorithm 1 beginning with the cycle C_8. By removing one by one and in the specified order the

Table 2
Adjacency list of the (4, 8)-cage.

$1 \to \{28, 55, 71, 72\}$	$2 \to \{28, 58, 68, 75\}$	$3 \to \{28, 61, 65, 78\}$	$4 \to \{29, 62, 66, 73\}$
$5 \to \{29, 56, 63, 76\}$	$6 \to \{29, 59, 69, 79\}$	$7 \to \{30, 57, 64, 74\}$	$8 \to \{30, 60, 70, 77\}$
$9 \to \{30, 54, 67, 80\}$	$10 \to \{31, 59, 67, 72\}$	$11 \to \{31, 62, 64, 75\}$	$12 \to \{31, 56, 70, 78\}$
$13 \to \{32, 54, 65, 73\}$	$14 \to \{32, 57, 71, 76\}$	$15 \to \{32, 60, 68, 79\}$	$16 \to \{33, 61, 69, 74\}$
$17 \to \{33, 55, 66, 77\}$	$18 \to \{33, 58, 63, 80\}$	$19 \to \{34, 60, 63, 72\}$	$20 \to \{34, 54, 69, 75\}$
$21 \to \{34, 57, 66, 78\}$	$22 \to \{35, 58, 70, 73\}$	$23 \to \{35, 61, 67, 76\}$	$24 \to \{35, 55, 64, 79\}$
$25 \to \{36, 56, 68, 74\}$	$26 \to \{36, 59, 65, 77\}$	$27 \to \{36, 62, 71, 80\}$	$28 \to \{1, 2, 3, 37\}$
$29 \to \{4, 5, 6, 37\}$	$30 \to \{7, 8, 9, 37\}$	$31 \to \{10, 11, 12, 38\}$	$32 \to \{13, 14, 15, 38\}$
$33 \to \{16, 17, 18, 38\}$	$34 \to \{19, 20, 21, 39\}$	$35 \to \{22, 23, 24, 39\}$	$36 \to \{25, 26, 27, 39\}$
$37 \to \{28, 29, 30, 40\}$	$38 \to \{31, 32, 33, 40\}$	$39 \to \{34, 35, 36, 40\}$	$40 \to \{37, 38, 39, 41\}$
$41 \to \{40, 42, 43, 44\}$	$42 \to \{41, 45, 46, 47\}$	$43 \to \{41, 48, 49, 50\}$	$44 \to \{41, 51, 52, 53\}$
$45 \to \{42, 54, 55, 56\}$	$46 \to \{42, 57, 58, 59\}$	$47 \to \{42, 60, 61, 62\}$	$48 \to \{43, 63, 64, 65\}$
$49 \to \{43, 66, 67, 68\}$	$50 \to \{43, 69, 70, 71\}$	$51 \to \{44, 72, 73, 74\}$	$52 \to \{44, 75, 76, 77\}$
$53 \to \{44, 78, 79, 80\}$	$54 \to \{9, 13, 20, 45\}$	$55 \to \{1, 17, 24, 45\}$	$56 \to \{5, 12, 25, 45\}$
$57 \to \{7, 14, 21, 46\}$	$58 \to \{2, 18, 22, 46\}$	$59 \to \{6, 10, 26, 46\}$	$60 \to \{8, 15, 19, 47\}$
$61 \to \{3, 16, 23, 47\}$	$62 \to \{4, 11, 27, 47\}$	$63 \to \{5, 18, 19, 48\}$	$64 \to \{7, 11, 24, 48\}$
$65 \to \{3, 13, 26, 48\}$	$66 \to \{4, 17, 21, 49\}$	$67 \to \{9, 10, 23, 49\}$	$68 \to \{2, 15, 25, 49\}$
$69 \to \{6, 16, 20, 50\}$	$70 \to \{8, 12, 22, 50\}$	$71 \to \{1, 14, 27, 50\}$	$72 \to \{1, 10, 19, 51\}$
$73 \to \{4, 13, 22, 51\}$	$74 \to \{7, 16, 25, 51\}$	$75 \to \{2, 11, 20, 52\}$	$76 \to \{5, 14, 23, 52\}$
$77 \to \{8, 17, 26, 52\}$	$78 \to \{3, 12, 21, 53\}$	$79 \to \{6, 15, 24, 53\}$	$80 \to \{9, 18, 27, 53\}$

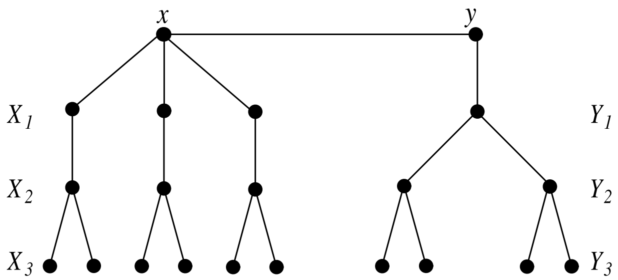

Fig. 5. Illustration of the non-existence in $\mathcal{F}^7_{20}$ of a graph with size 26 and $\Delta \geq 4$.

vertices of the set {1, 28, 37, 40, 41, 44, 51, 72, 2, 75, 52, 3, 78, 53, 10, 11, 31, 12, 38, 19, 20, 34, 21, 39, 29, 4, 62, 66, 73, 5, 56, 63, 76} from the (4, 8)-cage (see Table 2) we have constructed the graphs which provide the given lower bounds when $47 \leq \nu \leq 79$.

Next we prove that these lower bounds really represent the exact value of $f_7(\nu)$ for $19 \leq \nu \leq 36$, $\nu \neq 30$.

$f_7(19) = 24$. An immediate consequence of the inequalities (1)–(3).

The proof is similar when $\nu \in \{21, 22, 24, 25, 26, 27, 28, 29, 30, 33, 34, 35, 36\}$.

$f_7(20) = 25$. We assume that there is a graph $G \in \mathcal{F}^7_{20}$ with 26 edges. Then, $\delta = 2$ and $\Delta \geq 3$. Since $f_7(18) = 22$ and $f_7(17) = 20$, two vertices of degree 2 cannot be adjacent and every vertex of degree 3 is adjacent to at most a single vertex of degree 2. If $\Delta \geq 4$, with a distinguished edge xy, we obtain that $\nu(G) \geq 2 + |X_1| + |X_2| + |X_3| + |Y_1| + |Y_2| + |Y_3| \geq 21$, as is illustrated in Fig. 5. Therefore, $\Delta = 3$ and $n_2 = 8$. Taking into account that each of the eight vertices of minimum degree must be adjacent to two different vertices of degree 3, we obtain that there are sixteen vertices of degree 3 in $V(G)$. This contradicts the order of G.

$f_7(23) = 30$. We assume that there exists a graph $G \in \mathcal{F}^7_{23}$ with 31 edges. It has $\delta = 2$ and $\Delta \geq 3$. Taking into account the known results $f_7(21) = 27$, $f_7(20) = 25$ and $f_7(18) = 22$, we obtain that two vertices of minimum degree cannot be adjacent, that every vertex of degree 3 is adjacent to at most a single vertex of degree 2 and that every vertex of degree 4 is adjacent to at most three vertices of degree 2. If $\Delta \geq 4$, with a distinguished edge xy, we obtain $\nu(G) \geq 2 + |X_1| + |X_2| + |X_3| + |Y_1| + |Y_2| + |Y_3| \geq 24$. Similarly, since $\nu(G) = 23$, no vertex of degree 3 can be only adjacent to other vertices of degree 3. Therefore, we have that $\Delta = 3$ and that every vertex of degree 3 is adjacent to exactly one vertex of degree 2. From this assertion and the equality $n_2 = 7$, it follows that G can only have fourteen vertices of degree 3, contradicting that $n_3 = 16$.

$f_7(31) = 46$. A graph $G \in \mathcal{F}^7_{31}$ with size 47 has average degree $\bar{d}(G) = 94/31$ and, from Theorem 2 [4], it follows that $\nu(G) \geq 32$.

$f_7(32) = 47$. We assume that there is a graph $G \in \mathcal{F}^7_{32}$ with size 48. Then, $2 \leq \delta \leq 3$. First we notice that $\delta \neq 3$, because otherwise the graph G is 3-regular and, according to [7], for $r \geq 3$ there exists no r-regular graph with even girth $g \geq 8$ and order $\nu_0(r, g) + 2$. Hence, $\delta = 2$. Since $f_7(29) = 42$, if there are in G two consecutive vertices of degree 2, then there is no other vertex of minimum degree in $V(G)$. Moreover, as $f_7(28) = 40$, $f_7(27) = 38$, $f_7(26) = 36$, $f_7(25) = 34$, $f_7(24) = 32$ and $f_7(23) = 30$ the removal of p vertices, with $3 \leq p \leq 9$, implies the removal of at least $2p$ edges. This means that G

has few vertices of degree 2. Under this condition and taking into account the order and the girth of G, we get that $\Delta = 4$ and that every vertex of degree 4 must be adjacent to exactly two vertices of degree 2 and to two vertices of degree 3. Considering also that $n_2 = n_4$, we obtain that every vertex of degree 2 has two vertices of degree 4 in its neighborhood. Taking into account the above described properties of the graph G and using a distinguished edge xy in $E(G)$, we obtain that $\nu(G) \geq 2 + |X_1| + |X_2| + |X_3| + |Y_1| + |Y_2| + |Y_3| \geq 33$. □

Theorem 7 asserts that the lower bounds on $ex(\nu; \{C_3, C_4, \ldots, C_7\})$ provided in [16] are the best possible for every order $\nu \leq 34$.

6. Conclusions

The aim of this paper is the determination of $ex(\nu; \{C_3, C_4, \ldots, C_n\})$ which represents the maximum number of edges in a $\{C_3, C_4, \ldots, C_n\}$-free graph with given order ν. We have provided the exact value of this extremal function for the following cases:

$$n = 5, \quad \nu \in \{13, 15, 16, \ldots, 25, 27, 28, \ldots, 41\};$$
$$n = 6, \quad \nu \in \{17, 18, \ldots, 28\};$$
$$n = 7, \quad \nu \in \{19, 20, \ldots, 29, 31, 32, \ldots, 36\}.$$

We also give lower and upper bounds when

$$n = 5, \quad \nu \in \{43, 44, \ldots, 61\};$$
$$n = 6, \quad \nu \in \{29, 30, \ldots, 49\};$$
$$n = 7, \quad \nu \in \{37, 38, \ldots, 79\}.$$

We would like to remark that, for each specific order ν for which the extremal function has been determined, the upper bound given by Theorem 2 [4] is, in general, strictly greater than the exact value of the extremal function. Therefore, we think that, for the other orders ν, the exact value of the corresponding extremal function is closer to the given lower bounds than to the upper ones.

Acknowledgements

This research was supported by the Ministry of Education and Science, Spain, and the European Regional Development Fund (ERDF) under project MTM2005-08990-C02-02; and under the Andalusian Government project ‘‘MaDiscA’’ P06-FQM-01649.

References

[1] E. Abajo, C. Balbuena, A. Diánez, New families of graphs without short cycles and large size, Discrete Appl. Math. 158 (2010) 1127–1135.
[2] E. Abajo, A. Diánez, Graphs with maximum size and lower bounded girth, preprint.
[3] E. Abajo, A. Diánez, Extremal graphs with fixed excess, preprint.
[4] N. Alon, S. Hoory, N. Linial, The Moore bound for irregular graphs, Graphs Combin. 18 (2002) 53–57.
[5] C. Balbuena, M. Cera, A. Diánez, P. García-Vázquez, On the girth of extremal graphs without shortest cycles, Discrete Math. 308 (23) (2008) 5682–5690.
[6] N. Biggs, Construction for cubic graphs with large size, Electron. J. Combin. 5 (1998).
[7] N. Biggs, T. Ito, Graphs with even girth and small excess, Math. Proc. Cambridge Philos. Soc. 88 (1980) 1–10.
[8] G. Chartrand, L. Lesniak, Graphs and Digraphs, 3rd edition, Chapman and Hall, London, UK, 1996.
[9] M. Downs, R.J. Gould, J. Mitchem, F. Saba, $(D; n)$-cages, Congr. Numer. 32 (1981) 179–193.
[10] D.K. Garnick, Y.H.H. Kwong, F. Lazebnik, Extremal graphs without three-cycles or four-cycles, J. Graph Theory 17 (1993) 633–645.
[11] D.K. Garnick, Y.H.H. Kwong, F. Lazebnik, Algorithmic search for extremal graphs of girth five, in: Y. Alavi, A. Schwenk (Eds.), Proceedings of the Seventh Quadrennial International Conference on the Theory and Applications of Graphs, WMU (1992), Volume 2, John Wiley and Sons, New York, 1995, pp. 697–709.
[12] D.K. Garnick, N.A. Nieuwejaar, Non-isomorphic extremal graphs without three-cycles or four-cycles, JCMCC 12 (1992) 33–56.
[13] W.F. McGee, A minimal cubic graph of girth seven, Canad. Math. Bull. 3 (1960) 149–152.
[14] P. Turán, On an extremal problem in graph theory, Math. Fiz. Lapok 48 (1941) 436–452.
[15] W.T. Tutte, Connectivity in Graphs, University of Toronto Press, 1966.
[16] J. Tang, Y. Lin, C. Balbuena, M. Miller, Calculating the extremal number $ex(\nu; \{C_3, C_4, \ldots, C_n\})$, Discrete Appl. Math. 157 (9) (2009) 2198–2206.
[17] P. Wang, G.W. Dueck, S. MacMillan, Using simulated annealing to construct extremal graphs, Discrete Math. 235 (2001) 125–135.