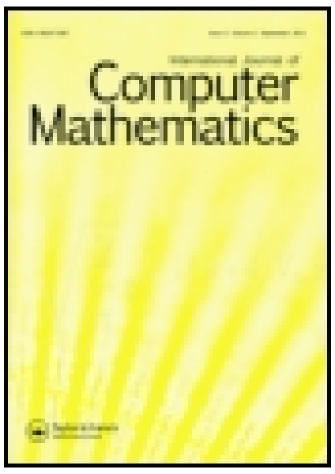


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# Stable matchings and stable partitions

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# STABLE MATCHINGS AND STABLE PARTITIONS*

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Recently Tan [7] defined a new structure for the stable roommates problem, called a "stable partition" which is a generalization of the notion of the stable matching. He proved that every instance of that problem contains at least one such structure, and obtained a succinct certificate of the non-existence of a stable matching. In this paper, we establish several results on properties of the stable partitions, present a simple proof of a recent theorem of Tan, and solve a maximum stable matching problem.

KEY WORDS: Stable roommates problem, stable matching, stable partition.

C.R. CATEGORIES: F2.2., G.2.1

## 1. INTRODUCTION

It is known that there may not exist a complete stable matching for a given instance of the stable roommates problem [1, 4, 6]. Tan [7] defined a new structure for this problem, called "a stable partition" which is a generalization of the notion of the complete stable matching, and proved that every instance of the stable roommates problem contains at least one such structure. In that paper, besides having an $O(n^2)$ algorithm to find a stable partition, he also obtained some interesting theoretical properties about the stable partitions. We mention the following results that are relevant to us.

i) There exists a stable partition for every instance of the stable roommates problem.
ii) Within a stable partition, the persons involved in the instance are partitioned into "even parties" and "odd parties". Any two stable partitions contain exactly the same odd parties.
iii) An instance of the stable roommates problem admits a complete stable matching if and only if there does not exist any "odd party".

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The structure of the stable partition is defined recently, and leads to some new results; it is therefore worthwhile to study some more properties about it. We observe that results (ii) and (iii) above are by products of the algorithm described in [7]; the proofs are quite long and complicated. In this paper, we estabish several results on properties of the stable partitions, give a simple proof of result (ii), and we shall explain that result (ii) implies (iii). Also we prove that, after "deleting" one person from an instance of the stable roommates problem, the number of odd parties in the resulting instance is either decreasing by 1 or increasing by 1. Applying this result, we define and solve the following problem: Find a maximum number of disjoint pairs of persons such that these pairs are stable among themselves (we call this a maximum stable matching problem). The time bound of our algorithm is $O(n^2)$.



## 2. DEFINITIONS

In this section, we state the stable roommates problem and give the definition of the stable partition introduced by Tan [7]. There is a set $S$ of $n$ people. Each person $i$ has a preference list that includes a subset $S_i$ of $S—\{i\}$, and a rank ordering (most preferred first) of the persons in $S_i$. For person $i$, the subset $S_i$ has the meaning that the only persons he is willing to be matched with are those in $S_i$. A preference relation $\mathscr{P}$ is defined to be a pair $(S, T)$, where $S$ is a set of $n$ persons, and $T$ is the table of preference list of these $n$ people. A complete matching $M$ is a partition of the $n$ persons into $n/2$ disjoint pairs of roommates such that for every pair $\{a, b\}$ in $M$, $a$ is on $b$'s list and $b$ is on $a$'s list. A complete matching $M$ is *unstable* if there are two persons who are not matched, but who each prefer the other to their respective mates in the matching; such a pair is said to *block* the matching $M$. A complete matching which is not unstable is called *stable*. The stable roomates problem, as it originally stated [1, 6], is to find a complete stable matching. Recently, Gusfield and Irving published a book [3] in which they listed over a hundred research papers related to this problem. Among those let us point out one, Irving [4] proposed an $O(n^2)$ algorithm to find one complete stable matching if there is one, or report that none exists. We note that there may not exist any complete stable matching. A stable partition is a structure that generalizes the notion of the complete stable matching; Tan [7] proved that every preference relation contains at least one such structure. We now begin to introduce it.

Let $T$ be a table of preference lists, if person $b$ is on the preference list of person $a$, then we write $(a|b)$ to denote the entry $b$ in $a$'s preference list. Define $r(a, b) = k$, if person $b$ occupies position $k$ in $a$'s preference list. If $r(a, b) < r(a, c)$, it means that person $a$ prefers $b$ to $c$. For the following definition, it is helpful to use the example given at the end as a running instance.

Let $(S, T)$ be a preference relation, and let $A$ be a subset of $S$. Denote $|A|$ the cardinality of set $A$. A cyclic permutation $\Pi(A) = \langle a_1, a_2, a_3, \ldots, a_k\rangle$ of the persons in $A$, where $k = |A|$. is called a *semi-party permutation* if one of the following three conditions holds:

i) $|A| \geqq 3$, $a_{i+1}$ and $a_{i-1}$ are on $a_i$'s preference list, and $r(a_i, a_{i+1}) < r(a_i, a_{i-1})$, $i = 1, 2, 3, \ldots, k$ (subscripts modulo $k$);

ii) $|A| = 2$, and $a_{i-1}$ is on $a_i$'s preference list, $i = 1, 2$ (subscript modulo 2);

iii) $|A| = 1$.

With a specified semi-party permutation $\Pi(A) = \langle a_1, a_2, \ldots a_k \rangle$ for persons in $A$, the entries in the preference lists of $A$ are classified into the following categories.

(I) If $|A| \geqq 3$, entry $(a_i|b)$ is said to be

i) a *superior* entry with respect to $\Pi(A)$, if $r(a_i, b) < r(a_i, a_{i-1})$;

ii) an *inferior* entry with respect to $\Pi(A)$, if $r(a_i, a_{i-1}) \leqq r(a_i, b)$; (Note: The inequality is "$\leqq$" not "$<$")

iii) a *party* entry with respect to $\Pi(A)$, if $b = a_{i+1}$ or $b = a_{i-1}$; for $i = 1, 2, 3, \ldots, k$ (subscripts modulo $k$).

(II) $|A| = 2$, *i.e.*, $k = 2$, $(a_i|b)$ is said to be

i) a *superior* entry with respect to $\Pi(A)$, if $r(a_i, b) < r(a_i, a_{i-1})$;

ii) an *inferior* entry with respect to $\Pi(A)$, if $r(a_i, a_{i-1}) < r(a_i, b)$; (Note: The inequality is "$<$" not "$\leqq$".)

iii) a *party* entry with respect to $\Pi(A)$, if $b = a_{i-1}$; for $i = 1, 2$ (subscripts modulo 2).

(III) If $|A| = 1$, then $(a_i|b)$ is a *superior* entry with respect to $\Pi(A)$ for every person $b$ on $a_i$'s preference list.

In the above definition, if there is no ambiguity, we will omit the word "with respect to $\Pi(A)$". For convenience, we will assume that the table of preference lists is *symmetric*, i.e., $a$ is on $b$'s list if and only if $b$ is on $a$'s.

Given a preference relation $(S, T)$, a *stable partition* $\Pi$ of $(S, T)$ consists of a partition of the set $S$; $S = \bigcup_{i=1}^{m} A_i$, $A_i \cap A_j = \phi$ if $i \neq j$, and a specified semi-party permutation $\Pi(A_i)$ for each $A_i$, $i = 1, 2, \ldots, m$, such that the following *stable condition* is satisfied:

If $(a|b)$ is a superior entry then $(b|a)$ is an inferior entry.

*Remark* If $T$ is not symmetric, then the above stable condition should be modified as follows: If $(a|b)$ is a superior entry then either $(b|a)$ is an inferior entry or $a$ is not on $b$'s preference list.

In the context of the above definition, the associated semi-party permutation $\Pi(A_i)$ is called a *party permutation* for $A_i$; and each $A_i$ is called a *party*. An *odd party* (*even party* respectively) is a party having odd (even respectively) cardinality. More precisely, these terms are defined with respect to the given stable partition $\Pi$. If there are ambiguities, we will say that $A_i$ is a party in $\Pi$ (or a $\Pi$-party), and $(a|b)$ is a superior entry in $\Pi$ (or a $\Pi$-superior entry), etc.

A stable partition $\Pi$ is specified by its party permutations and will be denoted by $\Pi = \{\Pi(A_1), \Pi(A_2), \Pi(A_3), \ldots, \Pi(A_m)\}$. Persons $a$ and $b$ are said to be a *matching pair* (or *matched*) in $\Pi$ if $\{a, b\}$ forms a 2-person party in $\Pi$. A subset $A$ of the all-person set $S$ is said to form a *party* (an odd party respectively), if there exists a stable partition $\Pi$ such that $A$ is a party (an odd party respectively) in $\Pi$.

We give the following example to illustrate the above definitions.

| person | | preference list | | | |
|---|---|---|---|---|---|
| 1 | . | 2 | . | 5 | . |
| 2 | . | 3 | . | 1 | . |
| 3 | superior | 4 | superior | 2 | inferior |
| 4 | . | 5 | . | 3 | . |
| 5 | . | 1 | . | 4 | . |
| 6 | | | superior | | |
| 7 | . | 8 | . | 10 | . |
| 8 | superior | 9 | superior | 7 | inferior |
| 9 | . | 10 | . | 8 | . |
| 10 | . | 7 | . | 9 | . |
| 11 | superior | | 12 | | inferior |
| 12 | . | | 11 | | |

A stable partition is shown above, there are four parties; namely $A_1 = \{1,2,3,4,5\}$,. $A_2 = \{6\}$, $A_3 = \{7,8,9,10\}$, $A_4 = \{11,12\}$ and $\Pi = \{\langle 1,2,3,4,5\rangle, \langle 6\rangle, \langle 7,8,9,10\rangle, \langle 11,12\rangle\}$. To complete this example, we just have to fill in all the other entries and follow the rule that whenever $(a|b)$ is a superior entry, then $(b|a)$ is inferior.

A preference relation may have more than one stable partition. We can identify at least two other stable partitions in the above example:

$$\Pi_1 = \{\langle 1,2,3,4,5\rangle, \langle 6\rangle, \langle 7,8\rangle, \langle 9,10\rangle, \langle 11,12\rangle\} \text{ and}$$

$$\Pi_2 = \{\langle 1,2,3,4,5\rangle, \langle 6\rangle, \langle 8,9\rangle, \langle 7,10\rangle, \langle 11,12\rangle\}.$$

As one can see that all these three stable partitions contain the same odd parties. Tan [7] proved that every preference relation contains at least on stable partition, and that any two stable partitions contain the same odd parties. Therefore the existence of an odd party depends on the preference relation, not on a particular stable partition.

As stated in [7], the notion of the stable partition generalized that of the complete stable matching in the following sense.

PROPOSITION 2.1 (Tan [7]) *A complete stable matching is a stable partition in which every party has cardinality two and vice versa.*

*Proof* This is directly from the definitions. $M = \{\{a_i, b_i\} \mid i = 1 \text{ to } n/2\}$ is a complete stable matching if and only if $\Pi = \{\langle a_i, b_i\rangle \mid i = 1 \text{ to } n/2\}$ is a stable partition. □

PROPOSITION 2.2 (Tan [7]) *A stable partition without any odd party induces a complete stable matching.*

*Proof* Suppose that $\Pi$ is a stable partition without any odd party. Let $A$ be an even party in $\Pi$ with party permutation $\langle a_1, a_2, a_3, \ldots, a_{2k}\rangle$, $k \geqq 2$. Then decomposing party $A$ into $k$ matching pairs $\langle a_1, a_2\rangle$, $\langle a_3, a_4\rangle, \ldots, \langle a_{2k-1}, a_{2k}\rangle$, we have a new stable partition

$$\Pi' = (\Pi - \{\langle a_1, a_2 \ldots, a_{2k}\rangle\}) \cup \{\langle a_1, a_2\rangle, \langle a_3, a_4\rangle, \ldots, \langle a_{2k-1}, a_{2k}\rangle\}.$$

This is because every superior entry in $\Pi'$ is a superior entry in $\Pi$, and every inferior entry in $\Pi$, other than the party entries, is an inferior entry in $\Pi'$. By continuing to decompose any even party having cardinality 4 or more, eventually we obtain a stable partition in which every party has cardinality two. $\square$

## 3. PROPERTIES OF STABLE PARTITIONS

In this section, we establish some properties about the stable partitions, and give a simple proof of result (ii) mentioned previously. Let $\mathscr{P} = (S, T)$ be a preference relation, and let $\Pi$ and $\Pi'$ be two stable partitions of $\mathscr{P}$. We shall study the relationship between $\Pi$ and $\Pi'$. Let $\mathscr{S}(\Pi, \Pi')$ be the set of all persons who have a superior entry in $\Pi$ as a $\Pi'$-party entry, and let $\mathscr{I}(\Pi, \Pi')$ be the set of all persons who have an inferior entry in $\Pi$ as a $\Pi'$-party entry. We note that $\mathscr{S}(\Pi, \Pi')$ and $\mathscr{I}(\Pi, \Pi')$ may not be disjoint. The following theorem says that, for each party $P$ in $\Pi$, the number of persons who have a superior entry in $\Pi$ as a $\Pi'$-party entry is equal to the number of persons who have an inferior entry in $\Pi$ as a $\Pi'$-party entry.

THEOREM 3.1 *Let $\mathscr{P} = (S, T)$ be a preference relation, and let $\Pi$ and $\Pi'$ be two stable partitions. Then*

*i)* $|\mathscr{S}(\Pi, \Pi')| = |\mathscr{I}(\Pi, \Pi')|$;

*ii)* $P \cap \mathscr{S}(\Pi, \Pi')| = |P \cap \mathscr{I}(\Pi, \Pi')|$ for every party $P$ in $\Pi$.

We remark that this result is an extension of the following theorem proved in [5]: Consider the stable marriage problem [3], where the people are divided into two sexes. If man $m$ and woman $w$ are partners in some stable matching $M$ then

i) there is no stable matching $M'$ in which both $m$ and $w$ have worse partner;

ii) there is no stable matching $M'$ in which both $m$ and $w$ have better partner.

Before proving Theorem 3.1, we need the following proposition:

PROPOSITION 3.2 *Let $(S, T)$ be a preference relation, and let $\Pi$ and $\Pi'$ be two stable partitions.*

*i) If a person has a $\Pi$-inferior entry as a party entry in $\Pi'$, then this person belongs to a $\Pi$-party of size 2 or more.*

*ii) Let $P = \langle a_1, a_2, \ldots, a_k\rangle$ be a party in $\Pi$ with $k \geqq 2$. If $a_i \in \mathscr{I}(\Pi, \Pi')$, then $a_{i-1}$ and $a_{i+1}$ are in $\mathscr{S}(\Pi, \Pi')$, (subscripts modulo $k$).*

*Proof* (i) This is trivial, because a person forming a $\Pi$-party by himself has no inferior entry in $\Pi$.

(ii) We consider two cases $k = 2$ and $k \geqq 3$.

*Case 1.* $k=2$ Suppose $a_1 \in \mathscr{I}(\Pi,\Pi')$. Then $(a_1|a_2)$ is superior in $\Pi'$, so $(a_2|a_1)$ is inferior in $\Pi'$. This implies that, in $a_2$'s list, there is an entry before $(a_2|a_1)$ which is a party entry in $\Pi'$. Hence $a_2 \in \mathscr{S}(\Pi,\Pi')$.

*Case 2.* $k \geqq 2$ Since $a_i$ has a $\Pi$-inferior entry as a party entry in $\Pi'$, and $(a_i|a_{i-1})$ is the first entry on $a_i$'s list which is $\Pi$-inferior, so $(a_i|a_{i+1})$ must be superior in $\Pi'$. Thus $(a_{i+1}|a_i)$ is inferior in $\Pi'$, and $a_{i+1} \in \mathscr{S}(\Pi,\Pi')$. As for $a_{i-1}$, since there is a $\Pi$-inferior entry on $a_i$'s list which is a party entry in $\Pi'$, we know that entry $(a_i|a_{i-1})$ is either a party entry in $\Pi'$ or a superior entry in $\Pi'$. Therefore $(a_{i-1}|a_i)$ is either a party entry or an inferior entry in $\Pi'$. In both cases $a_{i-1} \in \mathscr{S}(\Pi,\Pi')$. □

We may now prove Theorem 3.1.

*Proof* We first claim that $|\mathscr{S}(\Pi,\Pi')| \leqq |\mathscr{I}(\Pi,\Pi')|$. We prove this by defining a one to one mapping $f$ from $\mathscr{S}(\Pi,\Pi')$ into $\mathscr{I}(\Pi,\Pi')$ as follows: For each person $a$ in $\mathscr{S}(\Pi,\Pi')$, we define $f(a)=b$, where $(a|b)$ is the first entry on $a$'s list (starting from the most preferred entry) which is a $\Pi'$-party entry. Since $a \in \mathscr{S}(\Pi,\Pi')$, entry $(a|b)$ is a superior entry in $\Pi$ and is also a party entry in $\Pi'$, so $(b|a)$ is an inferior in $\Pi$ and is a party entry in $\Pi'$. This implies that $b \in \mathscr{I}(\Pi,\Pi')$. Suppose that $f$ is not one to one, then there exist two distinct persons $a$ and $a'$ in $\mathscr{S}(\Pi,\Pi')$ such that $f(a)=f(a')=b$. This means that $(a|b)$ ($(a'|b)$ respectively) is the first entry on $a$'s list ($a'$'s list respectively) which is a party entry in $\Pi'$. However, both $(b|a)$ and $(b|a')$ are party entries in $\Pi'$, so one of them is $\Pi'$-superior, without loss of generality, say $(b|a)$. Then $(a|b)$ is inferior in $\Pi'$, there has to be another entry on $a$'s list before $(a|b)$ which is a $\Pi'$-party entry, given a contradiction, thus $|\mathscr{S}(\Pi,\Pi')| \leqq |\mathscr{I}(\Pi,\Pi')|$.

We now prove that $|P \cap \mathscr{S}(\Pi,\Pi')| \geqq |P \cap \mathscr{I}(\Pi,\Pi')|$ for every party $P$ in $\Pi$. For a single person party in $\Pi$, this is trivial, because this person does not have any $\Pi$-inferior entry. Let $P=\langle a_1,a_2,\ldots,a_k\rangle$ be a $\Pi$-party with $k \geqq 2$. If $a_i \in \mathscr{I}(\Pi,\Pi')$, by Proposition 3.2, $a_{i-1}$ and $a_{i+1}$ are in $\mathscr{S}(\Pi,\Pi')$. Then the mapping defined by $g(a_i)=a_{i+1}$ (or alternatively $g'(a_i)=a_{i-1}$) is one to one from $|P \cap \mathscr{I}(\Pi,\Pi')|$ into $|P \cap \mathscr{S}(\Pi,\Pi')|$. So $|P \cap \mathscr{I}(\Pi,\Pi')| \leqq |P \cap \mathscr{S}(\Pi,\Pi')|$. Then we have

$$|\mathscr{I}(\Pi,\Pi')| = \sum_{P \text{ is a } \Pi\text{-party}} |P \cap \mathscr{I}(\Pi,\Pi')|$$

$$\leqq \sum_{P \text{ is a } \Pi\text{-party}} |P \cap \mathscr{S}(\Pi,\Pi')| = |\mathscr{S}(\Pi,\Pi')|.$$

Observing $|\mathscr{S}(\Pi,\Pi')| \leqq |\mathscr{I}(\Pi,\Pi')|$, we conclude that $\mathscr{S}(\Pi,\Pi') = |\mathscr{I}(\Pi,\Pi')|$ and $|P \cap \mathscr{S}(\Pi,\Pi')| = |P \cap \mathscr{I}(\Pi,\Pi')|$ for every party $P$ in $\Pi$. □

The following result, in a way similar to Proposition 3.2, can be proved from the above theorem.

COROLLARY 3.3 *Let $(S,T)$ be a preference relation, and let $\Pi$ and $\Pi'$ be two stable partitions.*

*i) If a person has a $\Pi$-superior entry as a party entry in $\Pi'$, then this person belongs to a $\Pi$-party of size 2 or more.*

*ii) Let $P=\langle a_1,a_2,\ldots,a_k\rangle$ be a party in $\Pi$ with $k\geqq 2$. If $a_i\in\mathscr{S}(\Pi,\Pi')$, then $a_{i-1}$ and $a_{i+1}$ are in $\mathscr{I}(\Pi,\Pi')$ (subscripts modulo $k$).*

*Proof* (i) For a single person party in $\Pi$, this particular person has no $\Pi$-inferior entry. So the result follows immediately from (ii) of Theorem 3.1.

(ii) If $a_{i+1}\notin\mathscr{I}(\Pi,\Pi')$ ($a_{i-1}\notin\mathscr{I}(\Pi,\Pi')$ respectively), then the mapping $g'$ ($g$ respectively) defined in the above Theorem 3.1 is one to one from $|P\cap\mathscr{I}(\Pi,\Pi')|$ into $|P\cap\mathscr{S}(\Pi,\Pi')|$, but not onto. This is because there is no element mapping to $a_i\in P\cap\mathscr{S}(\Pi,\Pi')$. Then $|P\cap\mathscr{I}(\Pi,\Pi')|<|P\cap\mathscr{S}(\Pi,\Pi')|$, given a contradiction. $\square$

Now, we can prove the following theorem in [7].

THEOREM 3.4 *Given a preference relation $(S,T)$, any two stable partitions contain exactly the same odd parties (not only having the same persons involved in a corresponding odd parties, but also with the same party permutation).*

*Proof* Let $\Pi$ and $\Pi'$ be two stable partitions. We first observe that $\Pi$ and $\Pi'$ contain the same odd parties of size 1. This is because $|P\cap\mathscr{S}(\Pi,\Pi')|=|P\cap\mathscr{I}(\Pi,\Pi')|$ for every party $P$ in $\Pi$, and a person constituting a single person odd party in $\Pi$ has no $\Pi$-inferior entry. So this particular person has to be a single person odd party in $\Pi'$.

Consider an odd party $P=\langle a_1,a_2,\ldots,a_{2k+1}\rangle$ in $\Pi$, where $k\geqq 1$. For each person $a_i$, he cannot form a single-person party in $\Pi'$. So, in $a_i$'s list, there must be an entry which is a $\Pi'$-party entry. We note that, being in a $\Pi$-party of size 3 or more, an entry in $a_i$'s list is either $\Pi$-superior or $\Pi$-inferior, thus $a_i$ is either in $\mathscr{S}(\Pi,\Pi')$ or in $\mathscr{I}(\Pi,\Pi')$.

If $a_i\in\mathscr{S}(\Pi,\Pi')$ ($a_i\in\mathscr{I}(\Pi,\Pi')$ respectively), by Corollary 3.3 (by Proposition 3.2 respectively), then $a_{i+1}\in\mathscr{I}(\Pi,\Pi')$ ($a_{i+1}\in\mathscr{S}(\Pi,\Pi')$ respectively) (subscript modulo $2k+1$). Since the number of persons in party $P$ is odd and the party permutation is cyclic, it is immediately that $a_i\in\mathscr{S}(\Pi,\Pi')$ if and only if $a_i\in\mathscr{I}(\Pi,\Pi')$. So, for each person $a_i$ in odd party $P$, he has two entries on his preference list as party entries in $\Pi'$; one is $\Pi$-superior and the other one is $\Pi$-inferior. Note that $(a_i|a_{i-1})$ is the first entry on $a_i$'s list which is $\Pi$-inferior. So, on $a_i$'s list, every entry before $(a_i|a_{i-1})$ is $\Pi'$-superior. In particular, $(a_i|a_{i+1})$ is $\Pi'$-superior, and hence $(a_{i+1}|a_i)$ is $\Pi'$-inferior, $i=1,2,\ldots,2k+1$, (subscripts modulo $2k+1$). Therefore, every entry after (including) $(a_i|a_{i-1})$ is $\Pi'$-inferior, $i=1,2,\ldots,2k+1$ (subscript modulo $2k+1$). This indicates that persons $\{a_1,a_2,\ldots,a_{2k+1}\}$ with the permutation $\langle a_1,a_2,\ldots,a_{2k+1}\rangle$ is also an odd party in $\Pi'$, and the theorem follows. $\square$

Let us relate some other works to the above results. The following theorems have been established in [1, 2 and 7] using various approaches; they are all immediate consequence of Theorem 3.4. We shall explain them respectively.

THEOREM 3.5 (Tan [7]) *Given a preference relation $(S,T)$, there exists a complete stable matching if and only if there does not exist any odd party.*

THEOREM 3.6 (Gale and Shapley [1]) *In the stable marriage problem, where the* *people are divided into two sexes of equal number, and where each person ranks all the members of the opposite sex in order of preference, then there always is a complete stable matching.*

Consider the following form of the stable marriage problem, where the number of males and the number of females may not be equal, and where the preference list of each person may include only a proper subset of the members of the opposite sex. Gale and Sotomayer [2] defined a stable matching, although in different terms, to be a stable partition in which every party has cardinality 2 or 1. They had the following result.

THEOREM 3.7 *If a person is unmatched (matched respectively) in some stable matching, then he/she is unmatched (matched respectively) in every stable matching.*

*Proof of Theorem 3.5* A complete stable matching is a stable partition without odd party. And a stable partition without odd party induces a complete stable matching. So, by Theorem 3.4, this result follows.

*Proof of Theorem 3.6* It is obvious that there does not exist any odd party of size more than 3 in this case. It is not difficult to see that there does not exist any odd party of size 1 either, since the number of males and the number of females are equal, and each person ranks all the members of the opposite sex. By Theorem 3.5, there exists a complete stable matching.

*Proof of Theorem 3.7* In this case, an unmatched person represents a single-person odd party. By Theorem 3.4, any two stable partitions contain the same odd parties. So the result obviously holds.

## 4. A MAXIMUM STABLE MATCHING

Given an instance of the stable roommates problem, since there may not exist any complete stable matching, let us consider the problem of finding a maximum number of disjoint pairs of persons such that these pairs are stable among themselves. Or in other words, we would like to exclude a minimum number of persons such that there is a complete stable matching for the remaining ones. We call this a maximum stable matching.

To achieve this goal, we shall prove the following two results. The reason for showing them is obvious: It is the existence of an odd party that prevents from having a complete stable matching.

i) After deleting a person who is in an odd party, the number of odd parties in the resulting instance is decreasing by one.

ii) After deleting an arbitrary person from the instance, the number of odd parties in the resulting instance can be decreasing by at most one. In fact, we shall prove that the number of odd parties in the resulting instance is either decreasing by 1 or increasing by 1.

First let us establish the above two facts, and explain later how one may use them to find a maximum stable matching.

Let $\mathscr{P}=(S,T)$ be a preference relation, and $a\in S$. We define the deletion of person $a$ from $\mathscr{P}$, denoted by $\mathscr{P}-a$, to be the preference relation $(S',T')$, where $S'=S-\{a\}$, and $T'$ is the preference lists obtained from $T$ by deleting the preference list of person $a$ and the entries $(x|a)$, for every $x\in S'$. In view of Theorem 3.4, any two stable partitions of $\mathscr{P}$ contain the same odd parties, we define $O.P.(\mathscr{P})$ to be the number of distinct odd parties in $\mathscr{P}$. Our first result is the following:

PROPOSITION 4.1 *Given a preference relation $\mathscr{P}=(S,T)$, let $\Pi$ be a stable partition of $\mathscr{P}$, and let $P=\langle a_1,a_2,\ldots,a_{2k+1}\rangle$ be an odd party with $k\geqq 0$. Then, after deleting $a_{2k+1}$,*

$$\Pi'=(\Pi-\langle a_1,a_2,\ldots,a_{2k-1}\rangle)\cup\{\langle a_1,a_2\rangle,\ \langle a_3,a_4\rangle,\ldots,\langle a_{2k-1},a_{2k}\rangle\}$$

*is a stable partition of $\mathscr{P}\quad a_{2k+1}$.*

*Proof* $\Pi'$ is simply obtained from $\Pi$ by decomposing the odd party $P=\langle a_1,a_2,\ldots,a_{2k+1}\rangle$ into $k$ matching pairs while excluding person $a_{2k+1}$ from the relation. It is clear that every superior entry in $\Pi'$ is a superior entry in $\Pi$, and every inferior entry in $\Pi$, other than the party entries, is an inferior entry in $\Pi'$. So $\Pi'$ is a stable partition of $\mathscr{P}-a_{2k+1}$. $\square$

In the context of the above proposition, we know that $O.P.(\mathscr{P}-a_{2k+1})=O.P.(\mathscr{P})-1$. One remark is required here: Even though the above proposition states the effect of deleting person $a_{2k+1}$ from $\mathscr{P}$, it makes no difference to delete any other person $a_i$, $i=1$ to $2k+1$, since the party permutation is cyclic. The following theorem says that, after deleting an arbitrary person from the preference relation, the number of odd parties is either decreasing by 1 or increasing by 1.

THEOREM 4.2 *Let $\mathscr{P}=(S,T)$ be a preference relation and let $a\in S$. Then $|O.P.(\mathscr{P}-a)-O.P.(\mathscr{P})|=1$.*

*Proof* Let $|S|=n$, $O.P.(\mathscr{P})=m$ and $O.P.(\mathscr{P}-a)=m'$. Then $|S-\{a\}|=n-1$, and both $n-m$ and $(n-1)-m'$ are even numbers. So the difference $|(n-m)-((n-1)-m')|=|m'-m+1|$ is also an even number. Hence $|O.P.(\mathscr{P}-a)-O.P.(\mathscr{P})|=|m'-m|$ is odd.

Suppose that there exists a person $a$ such that $|O.P.(\mathscr{P}-a)-O.P.(\mathscr{P})|=k$ and $k\neq 1$. Then by Proposition 4.1 and Theorem 3.4, $a$ cannot be in an odd party in the preference relation $\mathscr{P}$. So $a$ must be in an even party, by using the same technique used in proving Proposition 2.2, we can prove that there exists a stable partition $\Pi$ and a person $b$ such that $\{a,b\}$ forms a two-person party in $\Pi$. It is clear that $\Pi-\{\langle a,b\rangle\}$ is a stable partition of the preference relation $(\mathscr{P}-a)-b$. So $O.P.(\mathscr{P}-a-b)=O.P.(\mathscr{P})$, and $|O.P.(\mathscr{P}-a)-O.P.(\mathscr{P}-a-b)|=k$. We have just showed that if there exists a person $a$ in the preference relation $\mathscr{P}$ such that $|O.P.(\mathscr{P})-O.P.(\mathscr{P}-a)|=k, k\neq 1$ then there exists another person $b$ in the preference relation $\mathscr{P}'=\mathscr{P}-a$ such that $|O.P.(\mathscr{P}')-O.P.(\mathscr{P}'-b)|=k, k\neq 1$. Replacing person $a$ and preference relation $\mathscr{P}$ by $b$ and $\mathscr{P}'$ $(=\mathscr{P}-a)$, we may obtain the same conclusion recursively. Since $S$ is a finite set, where $\mathscr{P}=(S,T)$, this would give a contradiction. So $|O.P.(\mathscr{P})-O.P.(\mathscr{P}-a)|=1$ for every person $a\in S$. $\square$

Now, let us explain how we find a maximum stable matching. We first apply, for example, the $O(n^2)$ algorithm described in Tan [7] to find a stable partition. If there are $m$ odd parties in the partition, by Theorem 4.2 and Theorem 3.5, at least $m$ persons have to be excluded from a maximum stable matching. Using the method described in Proposition 4.1, we delete one person from each odd party to eliminate all the odd parties. The remaining table of preference lists does not contain any odd party, then we may apply the technique used in proving Proposition 2.2 to obtain a complete stable matching for the remaining persons which gives a maximum stable matching.

As for the time bound of this algorithm for finding a maximum stable matching, it is obvious that the main work is to find a stable partition, which is $O(n^2)$ as described in [7]. The time required for all the other work is dominated by that, so the overall time bound is $O(n^2)$.

COROLLARY 4.3 *Given a preference relation $\mathscr{P}=(S,T)$, the number of pairs in a maximum stable matching is equal to $\frac{1}{2}(n-O.P.(\mathscr{P}))$, where $n=|S|$ and $O.P.(\mathscr{P})$ is the number of odd parties in $\mathscr{P}$.*

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