Linear-algebraic λ-calculus: higher-order and confluence.

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5.12.06, Q-Day III, University of Glasgow

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Roadmap

1. Language & semantics
2. Confluence
3. Future works
4. Extras

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A language with

Computation

$\mathbf{t} ::= \mathbf{x} \mid \lambda \mathbf{x}.\mathbf{t} \mid (\mathbf{t}\,\mathbf{t}) \mid$

$\lambda \mathbf{x}.\mathbf{t}\,\mathbf{u} \longrightarrow \mathbf{t}[\mathbf{u}/\mathbf{x}]$ (B)

Linear algebra.

$\mathbf{t}+\mathbf{t} \mid \alpha.\mathbf{t} \mid \mathbf{0}$

$1.\mathbf{u} \longrightarrow \mathbf{u},\ 0.\mathbf{u} \longrightarrow \mathbf{0},\ \alpha.\mathbf{0} \longrightarrow \mathbf{0},$
$\mathbf{u}+\mathbf{0} \longrightarrow \mathbf{u},\ \alpha.(\beta.\mathbf{u}) \longrightarrow \alpha\times\beta.\mathbf{u},$
$\alpha.(\mathbf{u}+\mathbf{v}) \longrightarrow \alpha.\mathbf{u}+\alpha.\mathbf{v}$ (E)

$\mathbf{u}+\mathbf{u} \longrightarrow (1+1).\mathbf{u},$
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$\alpha.\mathbf{u}+\beta.\mathbf{u} \longrightarrow (\alpha+\beta).\mathbf{u}$ (F)

$\mathbf{t}\,(\mathbf{u}+\mathbf{v}) \longrightarrow (\mathbf{t}\,\mathbf{u})+(\mathbf{t}\,\mathbf{v}),$
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A minimal language with

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Linearity

As such this is not even linear:

$$\lambda \mathbf{x}.(\mathbf{x}\,\mathbf{x})(\mathbf{u}+\mathbf{v}) \longrightarrow^* \mathbf{u}\,\mathbf{u}+\mathbf{u}\,\mathbf{v}+\mathbf{v}\,\mathbf{u}+\mathbf{v}\,\mathbf{v}$$

Instead of

$$\lambda \mathbf{x}.(\mathbf{x}\,\mathbf{x})\mathbf{u}+\lambda \mathbf{x}.(\mathbf{x}\,\mathbf{x})\mathbf{v} \longrightarrow \mathbf{u}\,\mathbf{u}+\mathbf{v}\,\mathbf{v}$$

Copying vs cloning. . .

We must restrict (B) to **base vectors**.

But who are they?

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Higher-order

Fixed points, quantum control, black-box algos...
Higher-order usually means

$$\lambda \mathbf{x}.\mathbf{t}\ \lambda \mathbf{y}.\mathbf{u} \longrightarrow \mathbf{t}[\lambda \mathbf{y}.\mathbf{u}/\mathbf{x}]$$

Hence base vectors are
- abstractions (i.e. terms of the form $\lambda \mathbf{y}.\mathbf{u}$);
- variables (by some kind of recurrence).

Machine description interpretation, LISP quotes...

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(*) b an abstraction or a variable.

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Linearity

. . . is a matter for confluence!

$$\begin{array}{l} \lambda x.(x\,x)(u+v) \not\longrightarrow^* u\,u + u\,v + v\,u + v\,v \\ \quad\downarrow \\ \lambda x.(x\,x)u + \lambda x.(x\,x)v \longrightarrow u\,u + v\,v \end{array}$$

The restriction forbids the above banch.
Is it now confluent?

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Infinity

Untyped λ-calculus + linear algebra $\Rightarrow \ldots$

$$\mathbf{Yb} \equiv \lambda \mathbf{x}.(\mathbf{b} + (\mathbf{x}\,\mathbf{x}))\,\lambda \mathbf{x}.(\mathbf{b} + (\mathbf{x}\,\mathbf{x}))$$

$$\mathbf{Yb} \longrightarrow \mathbf{b} + \mathbf{Yb}$$

But whoever says infinity says trouble says... indefinite forms. These are again a matter for confluence!

$$\mathbf{Yb} - \mathbf{Yb} \longrightarrow \mathbf{b} + \mathbf{Yb} - \mathbf{Yb} \longrightarrow \mathbf{b}$$

$$\downarrow_*$$

$$\mathbf{0}$$

High school teacher says we must restrict (F) to **finite vectors**. But who are they?

Infinity

Untyped λ-calculus + linear algebra $\Rightarrow\ \infty$

$$\mathbf{Yb} \equiv \lambda \mathbf{x}.(\mathbf{b} + (\mathbf{x}\,\mathbf{x}))\,\lambda \mathbf{x}.(\mathbf{b} + (\mathbf{x}\,\mathbf{x}))$$
$$\mathbf{Yb} \longrightarrow \mathbf{b} + \mathbf{Yb}$$

But whoever says infinity says trouble says... indefinite forms. These are again a matter for confluence!

$$\mathbf{Yb} - \mathbf{Yb} \longrightarrow \mathbf{b} + \mathbf{Yb} - \mathbf{Yb} \longrightarrow \mathbf{b}$$
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. . . could be:

- finite vector norm? hard to define.
- normalizable terms? undecidable.
- normal terms? OK.

Comp. Sci "normal" = Maths "normal" in this calculus!

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Hiding indefinite forms

We restrict (F) to $\mathbf{u}$ normal.
Now

$$\mathbf{Yb} - \mathbf{Yb} \not\longrightarrow \mathbf{0}$$

But

$$\begin{array}{l} \lambda \mathbf{x}.(\mathbf{x}\,\mathbf{y} - \mathbf{x}\,\mathbf{y})\,\lambda \mathbf{y}.\mathbf{Yb} \longrightarrow^{*} \mathbf{0} \\ \quad \downarrow_{*} \\ \mathbf{Yb} - \mathbf{Yb} \end{array}$$

Hiding indefinite forms

We restrict (F) to $\mathbf{u}$ closed normal.
Now

$$\lambda \mathbf{x}.(\mathbf{x}\,\mathbf{y} - \mathbf{x}\,\mathbf{y})\lambda \mathbf{y}.\mathbf{Yb} \not\longrightarrow^* \mathbf{0}$$

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We restrict (A) to $\mathbf{u}$ closed normal, $\mathbf{u} + \mathbf{v}$ closed normal.

A minimal language... and its semantics.

Higher-order computation

$\mathbf{t} ::= \quad \mathbf{x} \mid \lambda \mathbf{x}.\mathbf{t} \mid (\mathbf{t}\,\mathbf{t}) \qquad \mid$

$\lambda \mathbf{x}.\mathbf{t}\,\mathbf{b} \longrightarrow \mathbf{t}[\mathbf{b}/\mathbf{x}]\,(*) \qquad (B)$

(*) $\mathbf{b}$ an abstraction or a variable.
(**) $\mathbf{u}$ closed normal.
(***) $\mathbf{u}+\mathbf{v}$ closed normal.
(****) $\mathbf{u}$ closed normal.

Linear algebra.

$\mathbf{t}+\mathbf{t} \mid \alpha.\mathbf{t} \mid \mathbf{0}$

$1.\mathbf{u} \longrightarrow \mathbf{u},\ 0.\mathbf{u} \longrightarrow \mathbf{0},\ \alpha.\mathbf{0} \longrightarrow \mathbf{0},$
$\mathbf{u}+\mathbf{0} \longrightarrow \mathbf{u},\ \alpha.(\beta.\mathbf{u}) \longrightarrow \alpha\times\beta.\mathbf{u},$
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$\mathbf{u}+\mathbf{u} \longrightarrow (1+1).\mathbf{u}\,(**)$
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Proof architecture.

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Non terminating. Define $B^{||}$.

Parallel moves lemma.

Terminating. AC extentions.

Critical pairs lemma.

-Automatic for non cond. N.

-By hands for F, A.

Parameterized on scalars.

Now the two commute.

The union is confluent.

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Roadmap

1. Language & semantics
2. Confluence
3. Future works
4. Extras

Motivations/Future works/Open problem

Quantum computation, computability in linear algebraic structure.

Typing: for unitarity? for developping novel logics.

Investigate the connection with linear λ-calculus, linear logic.

Open problem: find a model.

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Encoding booleans

$$\mathbf{true} \equiv \lambda x.\lambda y.x$$
$$\mathbf{false} \equiv \lambda x.\lambda y.y$$

Encoding the Not gate

$$\mathbf{Not} \equiv \lambda y.\Big((y * \mathbf{false}) * \mathbf{true}\Big)$$

Encoding the Phase gate

$$\mathbf{Phase} \equiv \lambda y.\left(\left(\left(y * \lambda x.(e^{i\frac{\pi}{4}}.\mathsf{true})\right) * \lambda x.\mathsf{false}\right) * \_\right)$$

where _ stands for dead code.

Running the Phase gate

Phase $*$ **true** yields

$$\lambda y.\Big(\big((y * \lambda x.(e^{i\frac{\pi}{4}}.\mathbf{true})) * \lambda x.\mathbf{false}\big) * \_\Big) * \mathbf{true}$$

$$\big((\mathbf{true} * \lambda x.(e^{i\frac{\pi}{4}}.\mathbf{true})) * \lambda x.\mathbf{false}\big) * \_$$

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Encoding the Hadamard gate

$$\textbf{Hadamard} \equiv \lambda y.\Big(\Big(\big(y * \lambda x.(\textbf{false} - \textbf{true})\big) * \lambda x.(\textbf{false} + \textbf{true})\Big) * \_\Big)$$

where _ stands for dead code.

Encoding tensors

$$\otimes \equiv \lambda x.\lambda y.\lambda f.((f * x) * y)$$

$$\pi_{\mathbf{1}} \equiv \lambda x.\lambda y.x$$

$$\pi_{\mathbf{2}} \equiv \lambda x.\lambda y.y$$

$$\bigotimes \equiv \lambda f.\lambda g.\lambda x.\Big(\Big(\otimes * (f * (\pi_1 * x))\Big) * (g * (\pi_2 * x))\Big)$$

E.g. $\mathbf{H}^{\otimes \mathbf{2}} \equiv ((\bigotimes \textbf{Hadamard}) * \textbf{Hadamard})$

Encoding the CNOT gate

$$\mathbf{Cnot} \equiv$$

$$\lambda x. \Bigg(\bigg(\otimes * (\pi_1 * x) \bigg) * \bigg(\Big((\pi_1 * x) * \big(\mathbf{Not} * (\pi_2 * x) \big) \Big) * (\pi_2 * x) \bigg) \Bigg)$$

Expressing Deutsch's algorithm parametrically

$$\mathbf{Deutsch} \equiv$$

$$\lambda f.\left(\mathbf{H}^{\otimes 2} * \left(f * \left(\mathbf{H}^{\otimes 2} * ((\otimes \mathbf{false}) * \mathbf{true})\right)\right)\right)$$

Trends

Van Tonder/ Valiron & Selinger's quantum λ-calculus: Uses the linear λ-calculus framework. Heterogeneous. Quantum resources (treated as linear) and classical resources (treated as nonlinear):

$$(\lambda_q) \qquad t ::= x \,|\, \lambda x.t \,|\, (t \;\; t) \,|\, c \,|\, !t \,|\, \lambda !x.t$$

$$c ::= 0 \,|\, 1 \,|\, H \,|\, CNOT \,|\, P$$

(together with the well-formedness rules of the classical linear λ calculus)

$$(\mathcal{R}_q) \qquad (\lambda x.t \;\; s) \xrightarrow{\beta} t[s/x]$$

$$H \;\; 0 \longrightarrow 0 + 1$$

$$H \;\; 1 \longrightarrow 0 - 1$$

$$\vdots$$

Term Rewrite Systems

These consist in

- a finite set of rules $l \longrightarrow r$, each interpreted as follows
- $t = t[\sigma l]_p$ should be rewritten into a term $t' = t[\sigma r]_p$.

Remarks:

- unambiguous so long as it is confluent. Preferably terminating.
- $l \longrightarrow r$ sometimes viewed as an oriented form of $l = r$.

E.g.

$$p \text{ plus } 0 = p \qquad p \text{ plus } S(q) = S(p) \text{ plus } q$$

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$$p \text{ plus } 0 \longrightarrow p \qquad p \text{ plus } S(q) \longrightarrow S(p) \text{ plus } q$$

Term Rewrite Systems

These consist in

- a finite set of rules $l \longrightarrow r$, each interpreted as follows
- $t = t[\sigma l]_p$ should be rewritten into a term $t' = t[\sigma r]_p$.

Remarks:

- unambiguous so long as it is confluent. Preferably terminating.
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$$2 + 2 \equiv S(S(S(S(0)))) \equiv 4$$

An algorithm

The algorithm for expressing vectors as linear combinations of the unknown:

- implemented elegantly as a TRS;
- shown terminating and confluent;
- provided a field, defines the notion of vectorial space.

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Let us seek a TRS for presenting numbers as linear combinations of unknowns:

We want to develop:

$$3.(\mathbf{x} + \mathbf{y}) \longrightarrow 3.\mathbf{x} + 3.\mathbf{y}$$

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But factor:

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$$3.(2.\mathbf{x}) \longrightarrow (3\times 2).\mathbf{x}$$

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We want to get rid of neutral elements:

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Modulo AC(+)

Consider:

$3.\mathbf{x} + 1.\mathbf{x} \longrightarrow (3+1).\mathbf{x}$

But: $3.\mathbf{x} + 1.\mathbf{x} \longrightarrow 3.\mathbf{x} + \mathbf{x}$

We need:

$$\lambda.\mathbf{u} + \mathbf{u} \longrightarrow (\lambda + 1).\mathbf{u}$$

An algorithm

$$\lambda.(\mathbf{u}+\mathbf{v}) \longrightarrow \lambda.\mathbf{u}+\lambda.\mathbf{v}$$
$$\lambda.\mathbf{u}+\mu.\mathbf{u} \longrightarrow (\lambda+\mu).\mathbf{u}$$
$$\lambda.(\mu.\mathbf{u}) \longrightarrow (\lambda\mu).\mathbf{u}$$
$$\mathbf{u}+\mathbf{0} \longrightarrow \mathbf{u}$$
$$1.\mathbf{u} \longrightarrow \mathbf{u}$$
$$0.\mathbf{u} \longrightarrow \mathbf{0}$$
$$\lambda.\mathbf{u}+\mathbf{u} \longrightarrow (\lambda+1).\mathbf{u}$$
$$\mathbf{u}+\mathbf{u} \longrightarrow (1+1).\mathbf{u}$$
$$\lambda.\mathbf{0} \longrightarrow \mathbf{0}.$$

Call it $\mathcal{R}$.

Definition: scalar rewrite system

These consist in terminating and confluent TRS on a language containing $+$, $\times$, 0, 1 and such that for all closed terms λ, μ, ν:

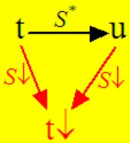
- $0 + \lambda$ and λ,
- $0 \times \lambda$ and 0,
- $1 \times \lambda$ and λ,
- $\lambda \times (\mu + \nu)$ and $(\lambda \times \mu) + (\lambda \times \nu)$,
- $(\lambda + \mu) + \nu$ and $\lambda + (\mu + \nu)$,
- $\lambda + \mu$ and $\mu + \lambda$,
- $(\lambda \times \mu) \times \nu$ and $\lambda \times (\mu \times \nu)$,
- $\lambda \times \mu$ and $\mu \times \lambda$

have the same normal forms and 0 and 1 are normal terms.

Termination and confluence properties

Proposition 1: Assuming $\mathcal{S}$ a confluent and terminating TRS for $\mathbb{K}$, then $\mathcal{R} \cup \mathcal{S}$ is a confluent and terminating TRS for $\mathbb{K}$-vectorial spaces.

Proof...


t
S*
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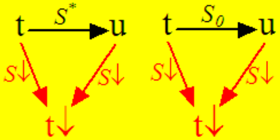
Def. *Subsumption*
A terminating and confluent TRS $\mathcal{S}$ subsumes a TRS $\mathcal{S}_0$ if whenever $t \xrightarrow{\mathcal{S}_0} u$, t and u have the same $\mathcal{S}$-normal form.
Consider $\mathcal{S}_0$:

$$0 + \lambda \longrightarrow \lambda$$
$$0 \times \lambda \longrightarrow 0$$
$$1 \times \lambda \longrightarrow \lambda$$
$$\lambda \times (\mu + \nu) \longrightarrow (\lambda \times \mu) + (\lambda \times \nu)$$

$\mathcal{S}$ subsumes $\mathcal{S}_0$.


t
S*
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S↓
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t↓

$\mathcal{R}$ terminates.

Proof technique:

Find $|.| :\longrightarrow \mathbb{N}$ such that

$t \longrightarrow r \quad \Rightarrow \quad |t| > |r|$. CiME does it for you.

$\mathcal{S}_0$ terminates.

Same method.

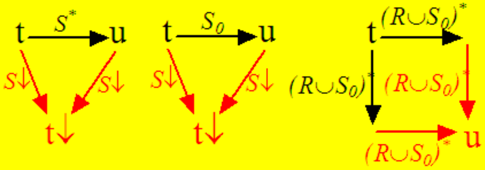
$\mathcal{R} \cup \mathcal{S}$ and $\mathcal{R} \cup \mathcal{S}_0$ terminate.

Proof: $\mathcal{S}$ and $\mathcal{S}_0$ leave $\mathcal{R}$'s polynomial interpretation unchanged.

$\mathcal{R} \cup \mathcal{S}_0$ is confluent.

Proof technique:

Since it is terminating it is enough to check that it is locally confluent. CiME does it for you.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S↓
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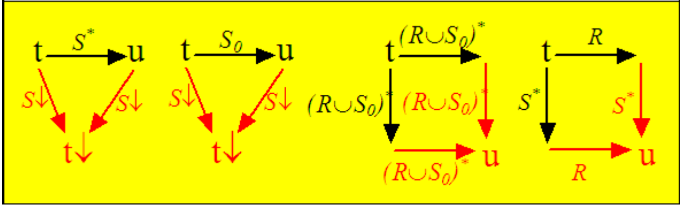
t
(R∪S0)*
(R∪S0)*
(R∪S0)*
u
(R∪S0)*

Commutation

A TRS $\mathcal{R}$ commutes with a TRS $\mathcal{R}'$ if whenever $t \xrightarrow{\mathcal{R}} u_1$ and $t \xrightarrow{\mathcal{R}'} u_2$, then there exists w such that $u_1 \xrightarrow{\mathcal{R}'} w$ and $u_2 \xrightarrow{\mathcal{R}} w$.

$\mathcal{R}$ commutes with $\mathcal{S}^*$.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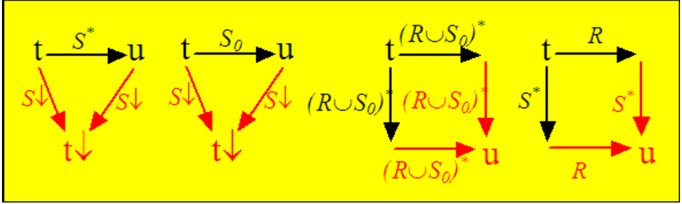
$(R\cup S_0)^*$
$(R\cup S_0)^*$
$(R\cup S_0)^*$
$(R\cup S_0)^*$
u
t
R
S^*
S^*
R
u

Key Lemma: Let R, S and S_0 be three TRS such that:

- S is terminating and confluent;
- S subsumes S_0;
- $R \cup S_0$ is confluent;
- R commutes with S^*;
- R terminates.

Then $R \cup S$ is confluent.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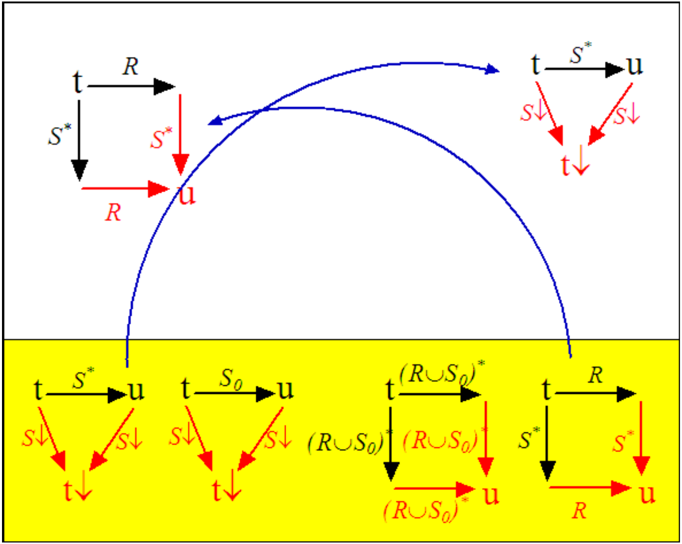
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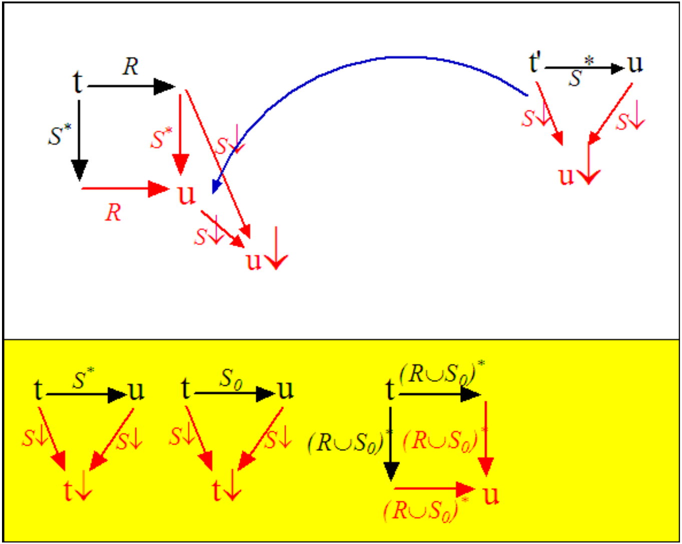
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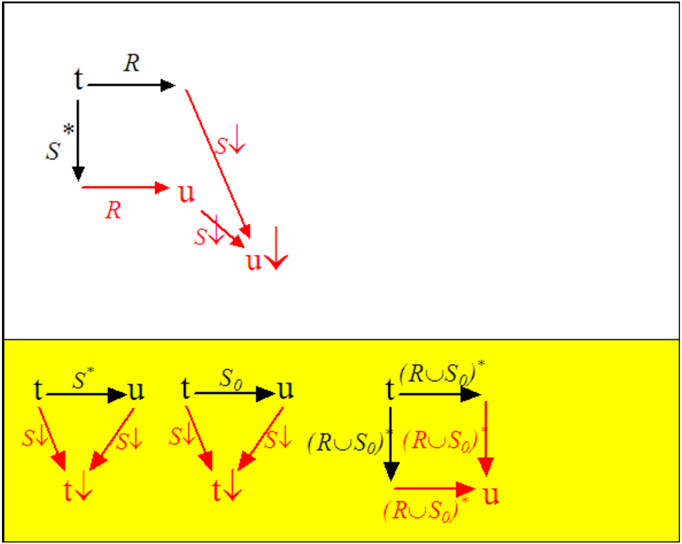
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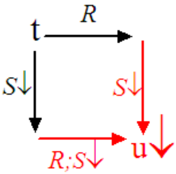
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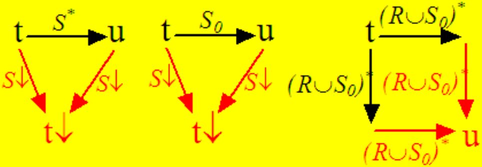
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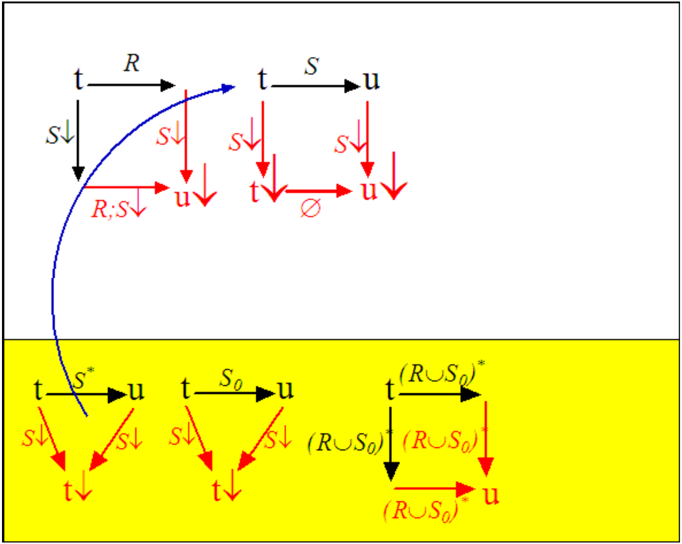
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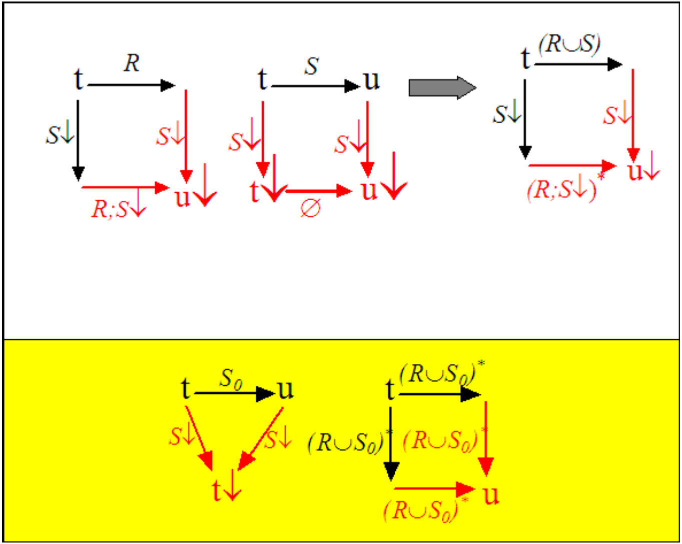
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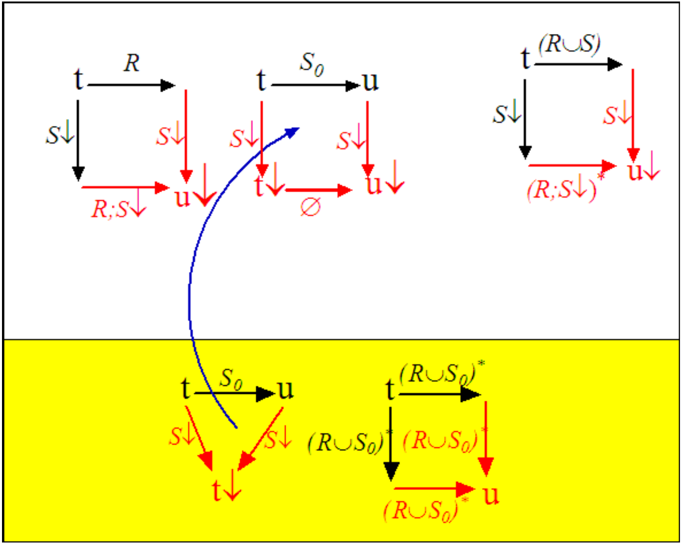

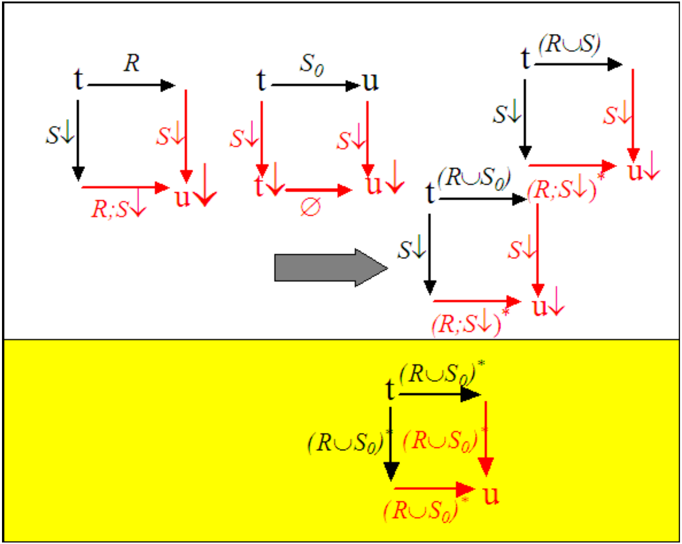
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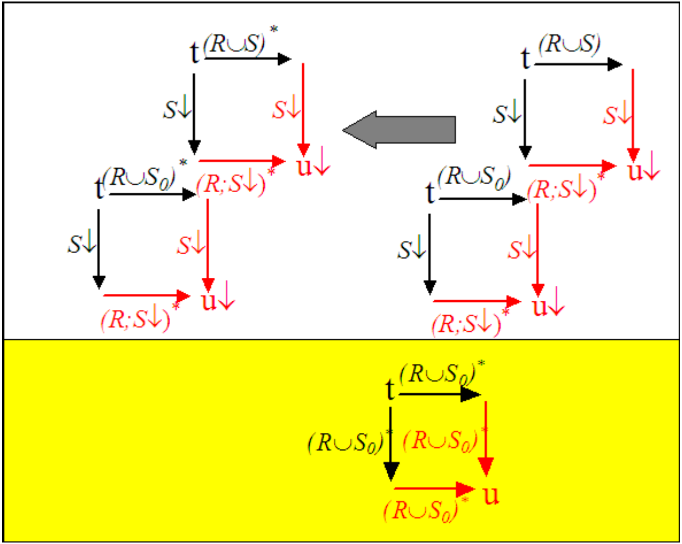

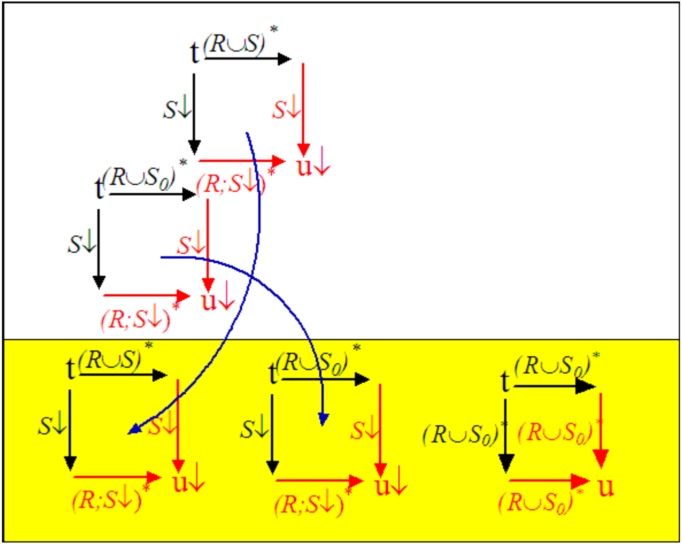
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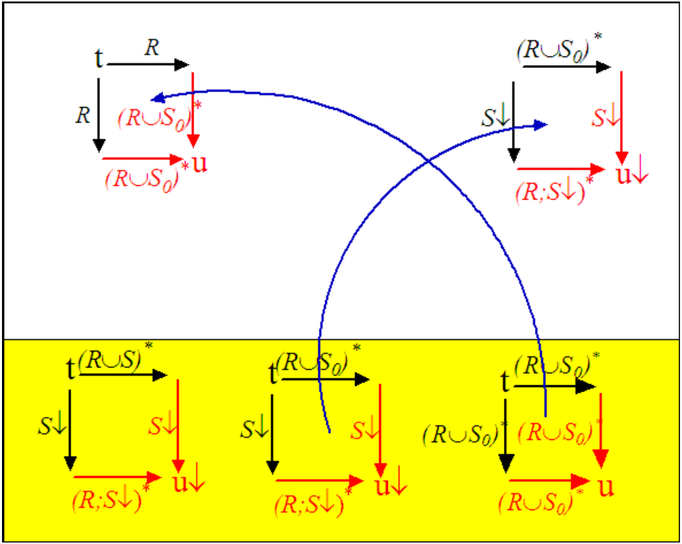
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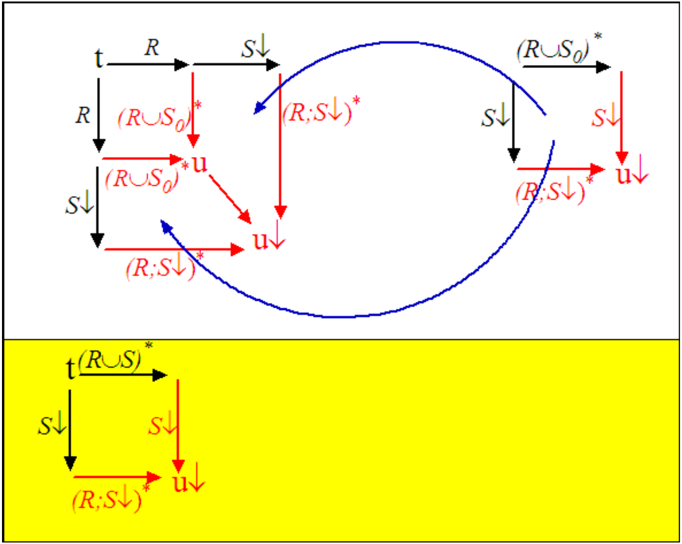
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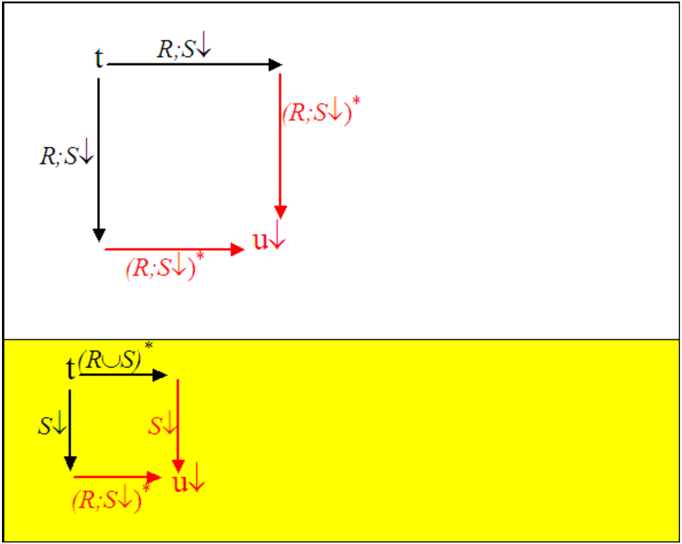
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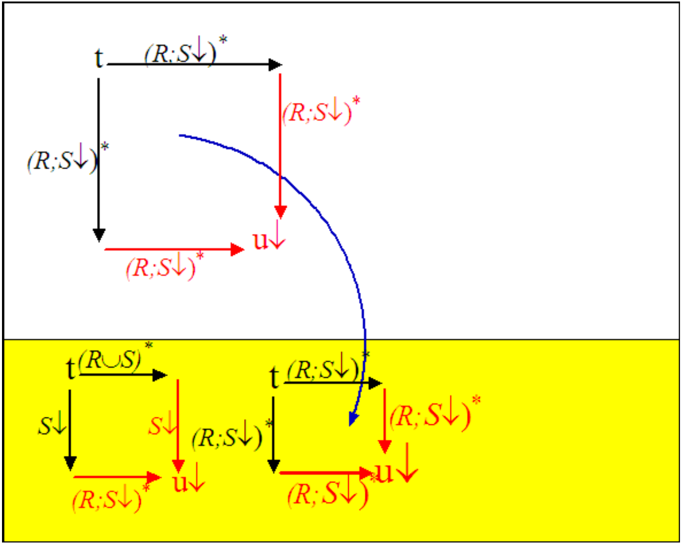
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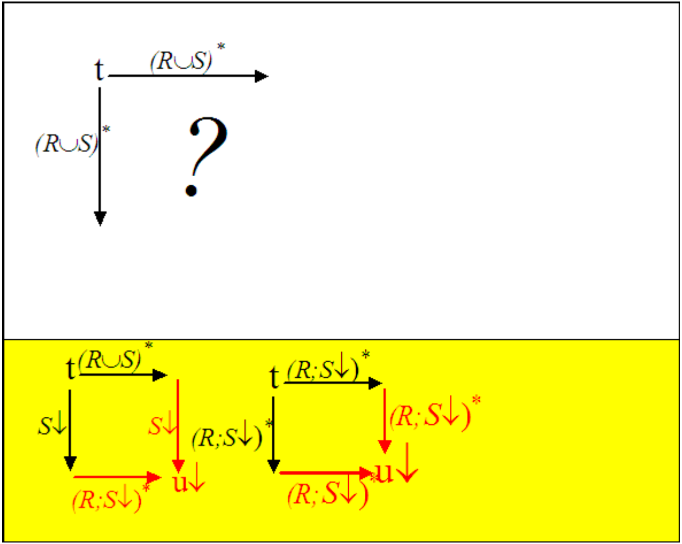
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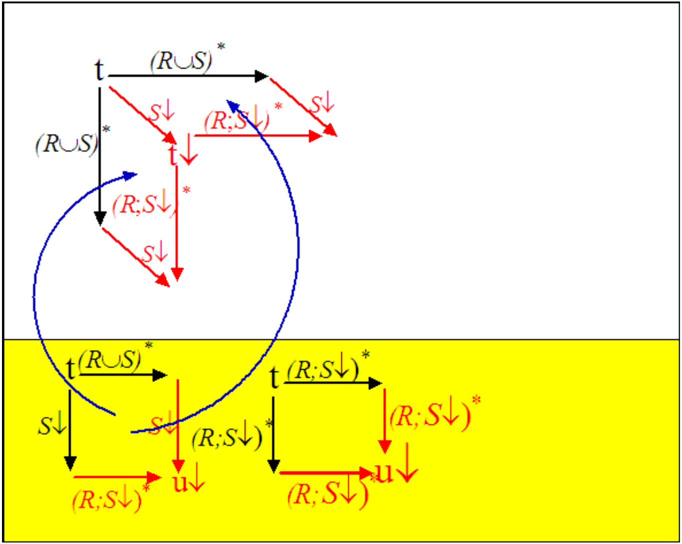
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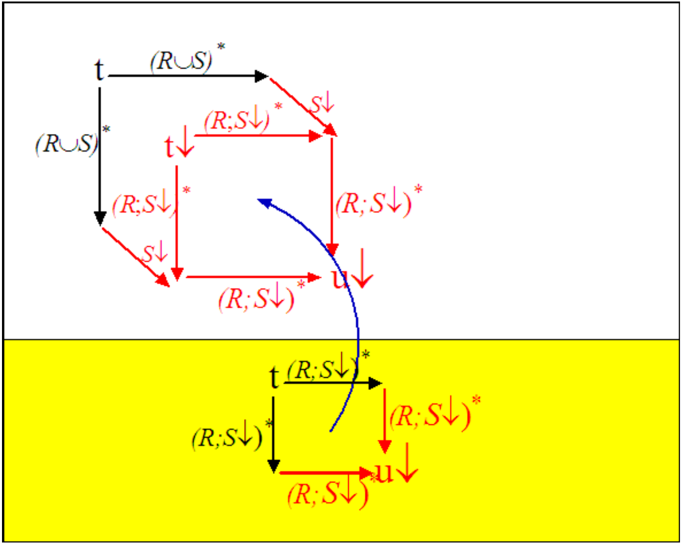
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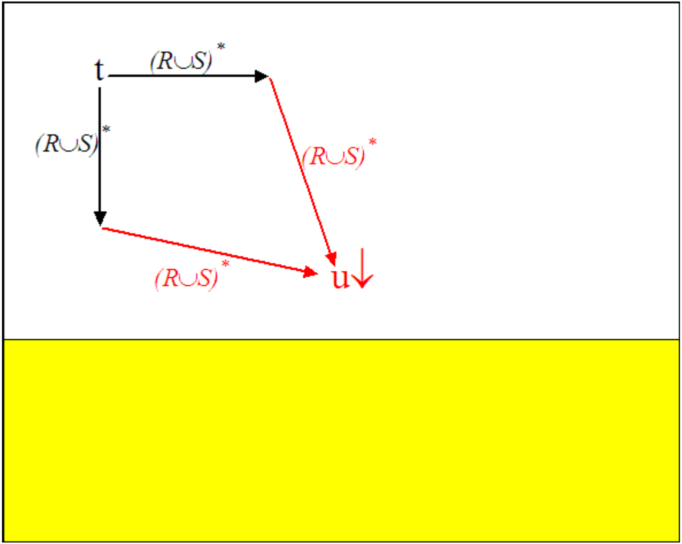
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The field of quantum computing is a. . .

[Solovay 95, Adleman et al. 97, Kitaev 97, Boykin et al.. . .]
A ring: the additive and multiplicative closure of diadic numbers (binary floats) together with $i^2 = -1$, $\frac{1}{\sqrt{2}} * \frac{1}{\sqrt{2}} = 1/2$.
I.e. a 4-dimensional module.

$$\mathbf{1} * \mathbf{v} \longrightarrow \mathbf{v}$$
$$\frac{\mathbf{1}}{\sqrt{\mathbf{2}}} * \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} \longrightarrow (1/2).\mathbf{1}$$
$$\frac{\mathbf{1}}{\sqrt{\mathbf{2}}} * \mathbf{i} \longrightarrow \frac{\mathbf{i}}{\sqrt{\mathbf{2}}}$$
$$\mathbf{i} * \mathbf{i} \longrightarrow -1.\mathbf{1}$$
$$\vdots$$